DOCUMENT RESUME

ED 092 407 SE 018 001

AUTHOR	Cox, Linda S.
TITLE	Analysis, Classification, and Frequency of Systematic Error Computational Patterns in the Addition, Subtraction, Multiplication, and Division Vertical Algorithms for Grades 2-6 and Special Education Classes.
INSTITUTION	Kansas Univ., Kansas City. Medical Center.
PUB DATE	Jun 74
NOTE	130p.
EDRS PRICE	MF-$0.75 HC-$6.60 PLUS POSTAGE
DESCRIPTORS	Addition; *Algorithms; Division; *Elementary School Mathematics; *Error Patterns; Multiplication; Number Concepts; *Research; Subtraction; *Whole Numbers
IDENTIFIERS	*Computation

ABSTRACT

Five reports from a 2-year study are presented. Frequencies and descriptions of systematic errors in the four algorithms in arithmetic were studied in upper-middle income, regular, and special education classrooms involving 744 children. Children were screened for adequate knowledge of basic facts and for receiving prior instruction on the computational process. Systematic errors contain a recurring incorrect computational process and are differentiated from careless errors and random errors. Errors were studied within levels of computational skill for each algorithm. Results showed that five to six percent of the children made systematic errors in the addition, multiplication, and division algorithms. The figure was 13 percent for the subtraction algortihm. One year later 23 percent of the children were making either the identical systematic error or another systematic error. (Author/JP)

Analysis, Classification, and Frequency of Systematic Error Computational Patterns in the Addition, Subtraction, Multiplication, and Division Vertical Algorithms for Grades 2-6 and Special Education Classes

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Note to the Reader

This research is presented in five separate reports. Each report was written so that it could be read separately and the reader would not have to refer to the other reports if he did not want to do so. The first report, the "Comparison" paper, presents summaries of all of the data and includes the data from the follow-up study one year later.

A detailed description of the systematic errors for each algorithm is presented in the tables of each respective report.

The Design and Rationale for the research is most thoroughly described in the "Comparison" paper and in the "Addition" paper, although they are summarized in each of the five reports. The literature review was kept specific for each report. References for each report will be found immediately following that report.

L. Cox

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Report No. 1

Comparison of Systematic-Error Computational Patterns in the Four Algorithms in Arithmetic Across Grades and Levels of Learning with Regular and Handicapped Children

Abstract

In a two-year study, frequencies and descriptions of systematic errors in the four algorithms in arithmetic were studied in upper-middle income, regular and special education classrooms involving 744 children. Children were screened for adequate knowledge of basic facts and for receiving prior instruction on the computational process. Systematic errors contain a reoccurring incorrect computational process and are differentiated from careless errors and random errors. Errors were studied within levels of computational skill for each algorithm. Results showed that 5-6% of the children made systematic errors in the addition, multiplication, and division algorithms. The figure was 13% for the subtraction algorithm. One year later 23% of the children were making either the identical systematic error or another systematic error.

Comparison of Systematic-Error Computational Patterns in the Four Algorithms in Arithmetic Across Grades and Levels of Learning with Regular and Handicapped Children[1]

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The Problem and Rationale

There has been limited specific research on systematic, careless, and random errors in the four computational algorithms in arithmetic. Because of the limited understanding of computational dysfunctions, the following questions were asked. Do systematic errors, as opposed to random or careless errors, occur frequently enough to merit the special attention of the classroom teacher? This is an important question because it assumes that if errors are systematic (reoccurring over and over and performed according to some unknown "rules"), then remediation would be possible. Should teachers for one stage of learning be more prepared to identify and remediate these errors than teachers at another level?

How persistent are systematic errors? Do children who make systematic errors still make them one year later? Do handicapped children make more and different systematic errors when compared with the systematic errors of regular classroom children? These questions became the focus of a two-year research study to analyze and determine the frequency of systematic errors in the four algorithms in arithmetic.

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[1] This study was supported by grant number NS 05362 from the National Institute of Neurological Diseases and Stroke to the Bureau of Child Research, University of Kansas.

This paper summarizes and compares the data from the four algorithms. The complete sets of data are presented in separate papers on addition, subtraction, multiplication, and division (Cox, 1973a, 1973b, 1973c, 1974).

Systematic Errors Defined

An error is labeled a systematic error when there is a repeatedly occurring incorrect response that is evident in a specific algorithmic computation. This incorrect process is apparent in three out of five problems of a given type. Systematic errors must be distinguished from random errors. Random errors give no evidence of a reoccurring incorrect process of thinking or recording.

Literature Review

A comprehensive review of the literature was conducted from 1900 through 1973 including a computer retrieval search of ERIC. The review focused on diagnosis, remediation, and error analysis in elementary school mathematics. There is limited research specifically on systematic errors for each algorithmic process.

Meyers (1924) called these errors "persistent" and was the first to document their occurrence in the literature. Grossnickle (1935, 1939), Brueckner and Elwell (1932), and Brueckner (1935) all examined various aspects of error analysis which included both systematic, random, and careless errors. Errors in those studies were categorized in general and broad categories of dysfunction.

Population

The sample consisted of 744 children from Johnson County, Kansas. The geographical setting is within the greater metropolitan Kansas City area. The white population of Johnson County numbers 215,845 and the non-white, 1031. The U. S. census (1970) indicated that 98% of the total labor force was employed, with a median family income of $13,384. The families with income below the poverty level was 2.9%. The median value of the owner-occupied homes was $22,000. Of the residents 25 years old, 79.6% have graduated from high school; 23.9% have college degrees; and the median number of school years completed was 12.8.

From the above described population, four public grade schools and two junior high schools in the Shawnee Mission Public Schools were selected. One Lutheran parochial school, one private elementary school, and two special education classrooms at the University of Kansas Medical Center were also selected. Schools in the sample were chosen on the basis of their willingness to participate, geographical location, and their number of available special education (handicapped) classrooms. The sample at each grade level was:

Total N = 744

Normal Population N = 564	Handicapped Population N = 180
2nd grade, N = 112	Primary, N = 45
3rd grade, N = 113	Intermediate, N = 70
4th grade, N = 116	Junior High, N = 65
5th grade, N = 110	
6th grade, N = 113	

The handicapped population consisted of pupils who had been placed in the Shawnee Mission Special Education classrooms and the classrooms at the University of Kansas Medical Center. Shawnee Mission classrooms consisted of Learning Problem classrooms and Educable Mentally Retarded classrooms. Pupils who were experiencing difficult progress in normal classrooms were placed in the Learning

Problems classroom if, in the judgment of the Shawnee Mission schools, they might benefit from special classroom placement. Pupils in the educable mentally retarded classrooms were usually at least two or more years retarded in language development. Classrooms at the University of Kansas Medical Center were labeled as classrooms for the emotionally disturbed.

Procedures

Levels of computational skill were identified in addition, subtraction, multiplication, and division. Eight levels were identified in addition; seven in subtraction; ten in multiplication; and ten in division. None of the levels were arranged in order of difficulty because research has not identified levels of difficulty for the algorithms. It is usually assumed that the more digits a child has to deal with, the more difficult the problem. The levels were organized by the number of digits, inclusion or exclusion of renaming, and the occurrence of zeros.

In order to complete the data collection within one year, data were gathered simultaneously for addition and subtraction, and then later for multiplication and division. Each week data sheets from one of the levels were distributed to the classrooms beginning in September, 1972. The classroom teachers administered the data sheets. Teachers were instructed that two requirements had to be met before a child's paper could be analyzed and included in the results. These requirements were:

1. The child had been taught the basic facts that pertain to the algorithm being studied. The child may not score 100% on a test of the facts, but he knows most of them.

2. The child has been exposed to the computational skill levels of the algorithm although he may not be proficient in its use.

If a child did not meet the above two requirements his paper was not included for analysis. If a child did meet these requirements his paper was analyzed and classified in only one of the following categories.

Systematic error. The child missed at least three out of five problems, recording repeatedly the same incorrect type of response in the algorithm.

Random error. The child missed at least three out of five problems but a pattern or systematic incorrect process could not be detected.

Careless error. The child missed only one or two problems out of five. He basically knows how to work the algorithm.

No error. All five problems are correct.

Incomplete data sheet. The child did not work all five problems so that the data sheets could not be classified as one of the above types. Great effort was made to keep this category as small as possible during the data collection.

Results

Table 1 shows the percentages of systematic errors for each grade level across algorithms and the number of papers analyzed for each skill level. A total of 11,763 papers was analyzed. The bottom line of the table indicates the average percentage of systematic errors for each algorithm for all grades and for both populations. The percentages in this line are very stable (5-6%) except for subtraction which is over twice as large (13%). Also, it can be noted from the table that grades 2 and 3 have a much higher occurrence of systematic errors than the percentages for grades 4, 5, and 6. Almost all instructional levels in the special education classrooms had high percentages except for addition in the junior high special education classroom (2%). Averages for special education students were over three times as high in multiplication and division as compared to children from regular classrooms. The addition algorithm produced the fewest systematic errors in the later stages of learning both for regular and special education. Inspection of Table 1 reveals that the percentages of systematic errors drop for each algorithm when you inspect the table from earliest grades to later grades.

Insert Table 1 (p. 14) here

Tables 2 and 3 illustrate representative examples of systematic errors for each algorithm. The tables are self explanatory.

Insert Tables 2 & 3 (pp. 15 & 16) here

There were a total of 223 systematic errors. The complete descriptions with illustrations (Cox, 1973a, 1973b, 1973c, 1974) are needed in order to develop specific training units to remediate these errors. However, for the sake of analyzation, the sheer number of these errors hinders understanding the dysfunction. Hence, a classification of these errors was made. Table 4 presents this classification.

Table 4 will be interpreted for only the addition algorithm. For the addition algorithm, 51 different systematic errors were grouped into four categories. One of these categories is the renaming concept. There were 23 systematic errors which occurred in different skill levels that could be grouped into the classification of dysfunction in renaming. However, the similarities in the 23 errors were so close that it may be quite possible that one training procedure will serve to correct all

23 of the errors.

Insert Table 4 (p.17) here

The categories in Table 4 will probably be familiar to most readers and detailed explanations will not be given except for the category of "concept of addition, subtraction, muliplication, and division." This category refers to all systematic errors which revealed confusion about the exact nature of the operation. For example, in multiplication 19 errors fell into this category. They included the following errors: no multiplication was performed and one of the factors was placed in the answer; one of the columns was omitted but the other columns correctly multiplied; and did not cross-multiply but instead multiplied each digit by the digit directly below it.

Twenty-one systematic errors were categorized as dysfunctions in the concept of division. These included: treating each digit of the dividend separately and not as a whole number; failing to perform subsequent operations of multiplication, subtraction, and formation of the next partial dividend; using the wrong operation to determine the partial dividend; division not performed in one of the columns of the dividend; and failure to indicate remainders as part of the division process.

Similar types of dysfunctions in addition were placed in the categories of "concept of addition." These included: adding the digits of the addends separately instead of treating the addends as two or three digit numbers; cross-adding such as adding a unit's digit to a tens digit; ignoring one of the columns but adding correctly in the other columns; not adding and using one of the addends for the answer.

For subtraction, the types of errors placed in the category of "concept of subtraction" were: answers which were larger than the minuend; using either the minuend or the subtrahend for the answer; not subtracting in one of the columns; and subtracted an extra number (such as 10) from the answer. In any of these cases, it is clear that the children do not understand the concepts underlying the operations.

Procedures for the Follow-Up Study

The question for study was, "Are systematic errors only a transient problem in learning or are they persistent?" To answer this question it was stated more specifically to read, "Do children who make systematic errors continue to make them one year later?" The answer to the last question is a qualified yes. Almost one-fourth of this sample did.

Selected specific skill levels in the subtraction and multiplication algorithms were selected for analysis in the follow-up study. One hundred ninety-one children who made systematic errors the preceding year were chosen. Both the children and skill levels were chosen without experimenter bias on any known variables.

Since a year had elapsed, almost all of the children had been assigned to new classrooms or in some cases, to new attendance centers. Every effort was made to locate these children. Of the 191 children 115 (60%) were located and tested in the follow-up study. Children were given identical data sheets that they had been given the preceding year.

Results

Table 5 shows the frequencies and percentages of the types of errors that were made one year later.

Insert Table 5 (p.18) here

Examining Table 5 indicates that 23% of the sample were making a systematic error one year later. Of this group, 16 (59% of the 23% or 14% of the entire sample) were making the same identical error one year later. The remaining 11 children in that group were making a different systematic error (41% of the 23% or 9% of the entire sample). In this latter group, the error was modified so that it was not identical nor similar to the previous error.

Discussion

A significant number of children are making systematic errors one year later. If instruction had been given it was not effective for retention of the concept. In many cases, instruction on the error may have never been given. There was no way to verify this. The significance of the results is that systematic errors are potentially long-term.

One could speculate that these errors need not be persistent because they could be amenable to instruction. This is particularly true in comparing those errors with random errors or careless errors. Since systematic errors contain a pattern with regard to the error the teacher can diagnose this error and begin remedial instruction on the error.

Summary

This two-year study analyzed the systematic errors in the four algorithms and determined that 5-6% of the children made systematic errors in addition, multiplication, and division algorithms. The figure is 13% for subtraction which is over twice as great as for the other algorithms. The percentages vary with grade levels. Second and third grade children produced the greatest frequencies in addition and subtraction. Third, fourth, and fifth grade children produced greater frequencies in multiplication and division than did sixth grade children. It is concluded that it is quite likely that all teachers will encounter children who make systematic errors. Assuming 30 children are assigned to the typical regular classroom, 5% of 30 is 1.5 so that it is quite likely a teacher will encounter such a child each year. Percentages were higher for all categories in special education classrooms. However, if typically 10 students are assigned to each class, then 17% of 10 = 1.7 and the chances are the same as for regular teachers.

Systematic errors can persist for at least a year as measured by this study. Almost one-fourth of the children in the follow-up study were making either the identical systematic error or another systematic error on the same algorithm one year later.

The descriptive research reported hereinbefore was done to analyze the types of errors that are made, to pinpoint the most frequently made systematic errors, and to establish a base for future research on remediation of these errors.

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Table 1

Comparison of the Percentages of Systematic Errors for All Grades Across All Four Algorithms

Grade	Addition	# of Papers	Subtraction	# of Papers	Multiplication	# of Papers	Division	# of Papers
2	10%	596	13%	382	*	*	*	*
3	6%	800	23%	544	6%	278	*	*
4	1%	816	8%	657	8%	725	7%	411
5	1%	208	6%	199	5%	872	5%	845
6	0%	194	6%	100	1%	847	3%	888
Averages for Grades 2-6	4%		11%		5%		5%	
Primary Sp. Ed.	5%	68	15%	33	*	*	*	*
Inter. Sp. Ed.	8%	454	24%	296	27%	206	21%	38
Jr. Hi. Sp. Ed.	2%	444	12%	340	11%	287	13%	194
Averages for Special Education	5%		17%		19%		17%	
Averages for All Grades, Both Populations	5%	3,580	13%	2,551	6%	3,229	6%	2,403

*Classrooms were not tested because no child could meet the requirements for the study.

Table 2

Representative Examples of Systematic Errors for the Addition and Subtraction Algorithm

Addition		Subtraction	
48 79 26 +3 +9 +7 15 25 15	Adds each digit separately; 4 + 8 + 3 = 15.	37 43 85 -4 -1 -3 13 32 52	Subtracted the single digit of the subtrahend from both the digits of the minuend.
48 79 26 +3 +9 +7 81 178 103	Adds the single digit addend to both of the digits in the other addend.	53 72 45 -14 -56 -19 41 24 34	Did not rename. Subtracted smaller minuend from larger subtrahend in the ones column.
52 19 14 86 27 45 +14 +73 +61 26 29 21	Adds each digit separately disregarding ones and tens column. (52 +86 +14 = 2 + 6 + 4 + 5 + 8 + 1 = 26)	$\overset{3}{\cancel{4}}\overset{19}{\cancel{9}}$ $\overset{1}{\cancel{2}}\overset{18}{\cancel{8}}$ $\overset{7}{\cancel{8}}\overset{13}{\cancel{3}}$ -11 -16 -32 218 12 411	Renamed the minuend when it was unnecessary. The difference in the ones column is a two digit number. The two-digit number is placed in the answer.
48 79 26 +3 +9 +7 411 718 213	Does not rename the sum of the ones column. This sum is placed in the answer and the digit in the tens column is placed to the left of the ones column.	$\overset{13}{\cancel{3}}7$ $\overset{14}{\cancel{4}}3$ $\overset{18}{\cancel{8}}5$ -4 -1 -3 93 132 152	Subtracted the single digit of the subtrahend from both digits of the minuend. Converted the tens column of the minuend into a two-digit number.
436 172 505 +11 +26 +74 547 398 1279	Adds correctly in the ones and tens column. The answer in the hundreds column is obtained by adding the digit in the hundreds column of the top addend to the digit in the tens column of the bottom addend. (e.g., 505 + 74 = 4 + 5 = 9; 7 + 0 = 7; 7 + 5 = 12; thus, answer is 1279.)	$4\overset{5}{\cancel{9}}\overset{13}{\cancel{3}}$ $3\overset{2}{\cancel{7}}\overset{16}{\cancel{6}}$ $2\overset{3}{\cancel{8}}\overset{15}{\cancel{5}}$ -45 -58 -39 418 338 206	Incorrectly renamed the minuend. The renamed number in the tens column is obtained by subtracting the smaller digit from the larger digit in the tens column. Subtraction in the tens column is then performed with this renamed number; e.g., the renamed ten of 493 - 45 is obtained by subtracting 9 - 4 = 5. Five is the renamed number.

Table 3

Representative Examples of Systematic Errors for the Multiplication and Division Algorithms

Multiplication	Division
1 2 2 47 16 29 x2 x4 x3 44 24 47 Multiplication is performed in the ones column. The "carried" number is multiplied by the tens digit of the multiplicand and this product is placed in the tens column of the answer.	713r.7/3 7)494 573r.6/4 6)342 Divided the divisor into the tens digit twice, once in the first division process and secondly as the single tens digit. (No work was shown on the child's paper). *Explanation:* 49 ÷ 7 = 7 34 ÷ 6 = 5 9 ÷ 7 = 1 44 ÷ 6 = 7 24 ÷ 7 = 3 22 ÷ 6 = 3 3 = r.7/3 4 = 4.6/4
47 16 29 x2 x4 x3 84 84 127 The renamed digit is multiplied instead of added to the product; example: 16 x4 4 (6 x 4 = 24) 8 (4 x 1 = 4); 84 (4 x renamed 2 = 8).	19 4)436 47 2)814 Errors occur because a zero is not placed in the tens place in the quotient. This occurs when a digit in the dividend is brought down and a division can not occur because the divisor is too large. Then the zero which should be placed in the quotient is omitted and the next division is computed. The result is quotients with zeros missing in the middle term.
1 2 2 47 16 29 x2 x4 x3 104 124 127 Added the "carried" digit before multiplying; e.g., 16 x 4 = 6 x 4 = 24; the renamed 2 is added to the 1 ten yielding a sum of 3 tens. 3 tens times 4 ones equals 12 tens. Thus, 124 is the answer.	403r.40 65)2835 260 23 0 235 195 40 402r.35 55)2345 220 14 0 145 110 35 Incorrectly placed the first digit of quotient which resulted in placing a zero in the tens column of the quotient.
47 13 38 x2 x5 x2 121 101 121 Reversal: "Carried" the wrong number when renaming the product of ones column; e.g., in 47 x 2, 7 x 2 = 14. The 1 was written in the ones column and the 4 was "carried" to the tens column.	

Table 4

Classification of Nature of Dysfunction Resulting in Systematic Errors for the Four Algorithms

Algorithm	No. of Different Systematic Errors	Algorithm	No. of Different Systematic Errors
Addition		Subtraction	
Renaming	23	Renaming	28
Concept of Addition	17	Concept of Subtraction	16
Wrong Operation	6	Wrong Operation	6
Place Value	5	Place Value	2
Total	51	Total	52
Multiplication		Division	
Concept of Multiplication	19	Concept of Division	21
Partial Products	13	Estimation	9
Mult. Process After Renaming	10	Partial Quotients	5
Add. Process After Renaming	7	Remainders	5
Renaming	6	Zeros in Quotients	5
Mult. with Zeros	6	Errors in Mult. or Sub.	4
Wrong Operation	4	Zeros in Dividend	2
Reversal of Digits	2	Partial Dividends	2
Total	67	Total	53

Table 5

Percentages and Frequency of Systematic Errors That Were Made Approximately One Year Later

Children making the same identical systematic error, approximately one year later.	16 (14%)
Children making a different systematic error, approximately one year later.	11 (9%)
	Subtotal = 27 (23%)
Children making a random error (missed 3 out of 5 problems, no pattern in the error).	11 (10%)
Children making a careless error (missed 1 or 2 out of 5 problems).	26 (23%)
Children making no error.	51 (44%)
	115 (100%)

Report No. 2

Systematic Errors in the Addition Algorithm Normal and Handicapped Populations[1]

Linda S. Cox[2]

The literature is replete with suggestions for the remedial teaching of arithmetic[3] but more actual research is needed on all of the various aspects of learning problems in arithmetic. This study was conceived to identify the most frequently occurring systematic errors in the addition, subtraction, multiplication, and division algorithms for whole numbers. In the process, a description of the systematic errors was developed for each algorithm. Only data from the addition algorithm is presented in this paper.

Research on the identification of systematic errors in computational processes was first cited by Myers (1924) in documenting the occurrence of "persistent" errors. Grossnickle (1939) first delineated random errors from systematic errors in the number facts. Applying this to the algorithms, a systematic error is a repeatedly occurring incorrect response in a specific algorithmic computation. This incorrect process will be evident in three out of five problems of a given type. The

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[1] This study was supported by grant number NS 05362 from the National Institute of Neurological Diseases and Stroke to the Bureau of Child Research, University of Kansas, for the study of Communication Disorders.

[2] The author wishes to acknowledge the contribution of Professor Lelon R. Capps who proposed the idea upon which this investigation is based.

[3] Glennon and Wilson (1972), Ashlock (1972), and Reisman (1972) are among the most recent of these writings.

systematic error is distinguished from a random error in that the random error gives no evidence of a repeatedly occurring incorrect process. The random error fails to show a pattern of incorrect thinking or recording. An example of a systematic error is:

$$\begin{array}{r} 24 \\ +3 \\ \hline 9 \end{array}$$

In this example the child correctly knows the addition facts but he adds each digit separately (2 + 4 + 3 = 9). A child making this error would make it in similar problems at least three out of five times before it is considered a systematic error.

It was established by Brueckner and Elwell (1932) that the uniformity in scoring and analyzing errors was reliable. There was a relatively high degree of consistency of the type of error found in pupil's work when at least three examples of a single type were solved incorrectly. This work was done with the algorithm for multiplying fractions.

Research Techniques

Burge (1934), in analyzing both random and systematic errors, reported the difference in detectable errors when they are analyzed using an interview technique compared to a paper and pencil test. The interview technique is necessary to determine how pupils arrive at the addition or multiplication facts. He reported that errors in combinations, repeating tables from a known combination, adding on to a lower combination, counting when carrying, and errors in carrying were detected most frequently in the interview technique. Most other errors were detectable from an analysis of test papers.

Analysis of paper and pencil data contains a certain subjective element but it does not affect the general conclusions. The type of analysis employed in this study is similar to techniques employed by Brueckner (1935), Brueckner and Elwell (1932), and Grossnickle (1935, 1939). The question of reliability of error has been studied by Grossnickle (1935) and Brueckner and Ellwell (1932). Conclusions are that a diagnosis made on the example of one incorrect response would be highly unreliable and that a minimum of three examples of a type of problem must be used before a reliable diagnosis can be made.

Population

Approximately 700 children in Johnson County, Kansas, participated in this study. This geographical area is within the greater metropolitan Kansas City region. The white population of Johnson County numbers 215,845 and the non-white, 1031. The U.S. census (1970) indicates that 98% of the total labor force was employed, with a median family income of $13,384. The families with income below the poverty level was 2.9%. The median value of the owner-occupied homes was $22,000. Of the residents 25 years old, 79.6% have graduated from high school; 23.9% have college degrees; and the median number of school years completed was 12.8.

From the above described population, four public grade schools and two junior high schools in the Shawnee Mission Public schools were selected. One Lutheran parochial school, one private elementary school, and two special education classrooms at the University of Kansas Medical Center were also selected. Schools in the sample were chosen on the basis of their willingness to participate, geographical location, and their number of available special education (handicapped) classrooms. The sample at each grade level was:

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The handicapped population consisted of pupils who had been placed in the Shawnee Mission Special Education classrooms and the classrooms at the University of Kansas Medical Center. Shawnee Mission classrooms consisted of Learning Problem classrooms and Educable Mentally Retarded classrooms. Pupils who were experiencing difficult progress in normal classrooms were placed in the Learning Problems classroom if it was in the judgment of the Shawnee Mission schools that they might benefit from special classroom placement. Pupils in the educable mentally retarded classrooms were usually at least two or more years retarded in language development. Classrooms at the University of Kansas Medical Center were labeled as classrooms for the emotionally disturbed.

Procedures

Levels of computational skill were identified in addition, subtraction, multiplication, and division. Eight levels were identified in addition; six in subtraction; 10 in multiplication; and 10 in division. Table 6 specifies the levels of skills for the addition algorithm. It should be noted that they were organized by the number of digits and the inclusion or exclusion of renaming.

Insert Table 6 (p.29) here

In order to complete the data collection within one year, data were gathered simultaneously for addition and subtraction, and then later for multiplication and division. Each week data sheets from one of the levels were distributed to the classrooms beginning in September, 1972. The classroom teachers administered the data sheets. Teachers were instructed that two requirements had to be met before a child's paper could be analyzed and included in the results. These requirements were:

1. The child had been taught the basic facts that pertain to the algorithm being studied. The child may not score 100% on a test of the facts, but he knows most of them.

2. The child has been exposed to the computational skill levels of the algorithm although he may not be proficient in its use.

If a child did not meet these two requirements his paper was not included in the results.

An example of a data sheet is shown in Figure 1. This example is from Level 4, Addition.

Insert Fig. 1 (p.43) here

Results

Table 7 shows the percentages of all types of errors across grade levels and for both populations combined. Overall, 5% of the population who met the requirements listed above made systematic errors in addition computation.

Insert Table 7 (p. 30) here

The percentage of the separate populations making systematic errors in addition is shown in Table 8. Table 8 should be read as follows: For Level 1 in addition, 9% of the 99 papers that were analyzed in grade 2, normal classrooms were classified as systematic errors; 2% in grade 3, normal classrooms; and 8% of the 57 papers that were analyzed in the intermediate handicapped classrooms were classified as systematic errors. Level 1 involved adding a two-digit plus a one-digit number with no renaming. The combined percentages of systematic errors for this skill level is 6% for both populations. The average number of papers analyzed per skill level is the same number as the average number of children per skill level who met the requirements for inclusion in the study.

Insert Table 8 (p.31) here

For the sake of brevity, tabulated results by classrooms are not presented, but it is important to report that in the normal populations each 2nd and 3rd grade classroom teacher had at least one child who made systematic errors in the addition algorithm. This was also true for each intermediate handicapped classroom teacher.

Descriptions and frequency of occurrence are presented in Tables 9 through 16 for each of the levels in the addition algorithm.

Insert Tables 9-16 (pp. 32-42) here

Discussion

Some of the errors were quite obvious but others were very difficult to analyze and took considerable time and effort. All of the systematic errors may not have been identified in this research but it is the opinion of the author that only a few would fall in this category. A great amount of time, consideration, and thoroughness was given to each analysis. Some of the errors are quite ingenious. In every case, the child's behavior indicates that he realizes patterns and structures are necessary for solving computational problems. He simply has not perceived or recorded the correct pattern. Each child participating in this study had been exposed to the correct process of solving the various levels of addition algorithms.

The various errors fall under the general categories of misconceptions regarding the nature of number, the nature of the addition operation, the function of place value, and the function of renaming. It is important to note that in almost every case, the addition facts were correct but the process involved in using the addition algorithm was wrong. This is important to note so that the teacher doesn't mistake the problem as a lack of knowledge of the addition facts and subsequently prescribes more drill on the facts. This is precisely what the child does not need.

It should be remembered that no child's work was included in this study unless he made the same error in three out of five problems of a given type. In 43% of the cases, or nearly half of the children who made systematic errors, did so in all five of the problems. They repeated the incorrect process in every problem that they worked. It should be noted that 28% of the children made systematic errors in four of the five problems and 29% made systematic errors in three out of five problems.

Conclusions

This research has shown that familiarity with systematic errors in computation is vitally important because each teacher will encounter at least one child who exhibits such behavior. Consider an analogy with the medical profession. Physicians must routinely deal with very common problems but they must be prepared also to treat the cases that occur at a frequency of 1 in 100 or more people. Likewise, we expect teachers to be able to identify and treat disorders in the learning process.

Future research should focus on the following questions: Is the occurrence of these errors related to Piagetian cognitive development? Were these children exposed to processes before they were ready? How persistent are these errors? What are efficient teaching procedures to correct these errors?

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Table 6

Levels* of Skill in Addition

Skill		Example
Level 1:	Adding a two-digit + one-digit number; no renaming.	23 +2
Level 2:	Adding a two-digit + one-digit number; with renaming.	18 +7
Level 3:	Adding a two-digit + two-digit number; no renaming.	43 +16
Level 4:	Adding a two-digit + two-digit number; with renaming.	19 +24
Level 5:	Adding a three-digit + a two-digit number; no renaming.	172 +26
Level 6:	Adding a three-digit + a two-digit number; renaming in the ones column only.	476 +17
Level 7:	Adding a three-digit + a two-digit number; renaming in the ones and tens column.	345 +76
Level 8:	Column addition--three two-digit numbers; with renaming.	46 39 +17

*The levels are not in order of increasing difficulty. They are organized by the number of digits and the inclusion or exclusion of renaming.

Table 7

All Types of Errors for Both Populations for All Grades in Addition Algorithm

	Systematic Errors	Random Errors	Careless Errors	No Errors	Incomplete Data Sheet
Skill Levels	3 out of 5	3 out of 5	1 or 2 out of 5		
1	6%	4%	16%	73%	1% = 100%
2	9%	5%	24%	61%	1% = 100%
3	1%	2%	12%	85%	0% = 100%
4	8%	5%	18%	68%	1% = 100%
5	2%	2%	15%	81%	0% = 100%
6	9%	6%	18%	66%	1% = 100%
7	3%	3%	17%	76%	0% = 100%
8	2%	2%	27%	68%	1% = 100%
Average	5%	4%	18%	72%	1% = 100%

Table 8

Percentage of Systematic Errors in Addition Algorithm by Population, Grade, and Skill Level

	Normal Populations					Handicapped Populations			Both Populations
	Grade								
Levels of Skill	2	3	4	5	6	Primary	Intermediate	Jr.Hi.	
1	9%	2%	*	*	*	0%	8%	*	6%
2	8%	21%	5%	*	*	0%	9%	2%	9%
3	1%	0%	0%	*	*	0%	1%	0%	0.4%
4	15%	12%	1%	*	*	0%	14%	2%	8%
5	1%	1%	0%	*	*	8%	5%	4%	2%
6	25%	7%	0%	*	*	22%	10%	0%	9%
7	*	4%	1%	2%	0%	*	12%	2%	3%
8	*	1%	2%	0%	0%	*	8%	2%	2%
Average % by Grade Level	10%	6%	1%	1%	0%	5%	8%	2%	5%
Total # of Papers Analyzed	596	800	816	208	194	68	454	444	3,580
Average # of Papers Analyzed/ Skill Level	99	100	102	104	97	11	57	63	

*Not tested at this grade level.

Table 9

Systematic Errors in Addition Algorithm: Level 1 - Adding a Two-Digit + One-Digit Number; No Renaming

Number of Errors		Error			
Normal	Handicapped				
8	0	46 +3 13	21 +8 11	15 +2 8	Adds each digit separately disregarding ones and tens column. 4 + 6 + 3 = 13.
0	7	46 +3 79	21 +8 109	15 +2 37	Adds single digit addend to both digits in the other addend. 3 + 6 = 9; 3 + 4 = 7; = 79.
1	2	46 +3 59	21 +8 39	15 +2 27	The number 1 is being carried to the tens column unnecessarily.
1	1	46 +3 43	21 +8 28	15 +2 12	The single digit addend is placed in the ones column in the answer. The digit in the tens column in the answer is selected from one of the digits in the top addend.
1	0	46 +3 43	21 +8 13	15 +2 13	The subtraction operation is performed instead of the addition operation.
0	1	46 +3 132	21 +8 168	15 +2 30	The multiplication operation is performed instead of the addition operation.
11	11				

Table 10

Systematic Errors in Addition Algorithm:
Level 2 - Adding a Two-Digit + One-Digit Number; With Renaming

Number or Errors			
Normal	Handicapped		Error
14	1	48 79 26 +3 +9 +7 411 718 213	Does not rename the sum of the ones column. This sum is placed in the answer and the digit in the tens column is placed to the left of the ones column.
8	0	48 79 26 +3 +9 +7 41 78 23	Didn't add the "carried" number.
4	3	48 79 26 +3 +9 +7 81 178 103	Adds the single digit addend to both of the digits in the other addend.
6	1	48 79 26 +3 +9 +7 45 70 19 or 21	Subtracts instead of adds.
4	0	48 79 26 +3 +9 +7 15 25 15	Adds each digit separately; 4 + 8 + 3 = 15.
1 37	0 5	48 79 26 +3 +9 +7 11 18 13	The digits in the ones column are added and placed in the answer. The tens column is ignored.

Table 11

Systematic Errors in Addition Algorithm:
Level 3 - Adding a Two-Digit + Two-Digit Number; No Renaming

Number of Errors			
Normal	Handicapped	Error	
1	0	$\begin{array}{r}43\\+16\\\hline 69\end{array}$ $\begin{array}{r}37\\+51\\\hline 98\end{array}$ $\begin{array}{r}76\\+22\\\hline 108\end{array}$	The number one is being "carried" to the tens column.
1 2	0 0	$\begin{array}{r}43\\+16\\\hline 17.\end{array}$ $\begin{array}{r}37\\+51\\\hline 38\end{array}$ $\begin{array}{r}76\\+22\\\hline 21\end{array}$	No addition is performed. The answer is either one more or one less than one of the addends.

Table 12

Systematic Errors in Addition Algorithm: Level 4 - Adding a Two-Digit + Two Digit Number; With Renaming

Number of Errors		Error	
Normal	Handicapped		
18	1	$\begin{array}{rrr}24&56&17\\+67&+28&+33\\\hline 81&74&40\end{array}$	Does not rename the ten from the sum of the ones column.
4	4	$\begin{array}{rrr}24&56&17\\+67&+28&+33\\\hline 811&714&410\end{array}$	The sum of the ones column is placed in the answer without renaming. The sum of the tens column is placed to the left of the ones column.
2	0	$\begin{array}{rrr}24&56&17\\+67&+28&+33\\\hline 82&75&41\end{array}$	Incorrectly renames. Places the number to be carried in the ones column and adds it to the sum of the ones column. (7 + 4 = 11 + renamed 1 = 12. Places 2 in the ones column. 6 + 2 = 8; = 82.)
1	0	$\begin{array}{rrr}24&56&17\\+67&+28&+33\\\hline 19&21&14\end{array}$	Adds the digits in the addends separately disregarding ones and tens column. (24 + 67 = 2 + 4 + 6 + 7 = 19).
2	1	$\begin{array}{rrr}24&56&17\\+67&+28&+33\\\hline 57&28&24\end{array}$	Subtracts instead of adds. If renaming is required in order to subtract in the ones column, then renaming is done. In the tens column, the smaller digit is subtracted from the larger one.
0	1.	$\begin{array}{rrr}12&15&11\\\cancel{2}4&\cancel{5}6&\cancel{1}7\\+67&+28&+33\\\hline 181&174&140\end{array}$	The renamed ten is "carried" to the hundreds column.

Table 12(continued)

0	1	24 +67 88	56 +28 79	17 +33 48	The sum in the ones column is one more than the largest digit in the ones column. Adds correctly in the tens column.
0 27	1 9	24 +67 101 90	56 +28 144 78	17 +33 50 50	The mechanics of the vertical multiplication algorithm are used, but instead of multiplying the digits, they are added. Only partial sums exist for an answer.

Table 13

Systematic Errors in Addition Algorithm:
Level 5 – Adding a Three-Digit + Two Digit Number; No Renaming

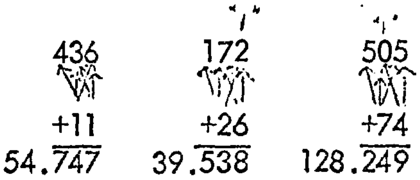
Number of Errors					
Normal	Handicapped				Error
1	2	436 +11 047	172 +26 098	505 +74 079	The hundreds column is not recorded in the answer.
1	0	436 +11 547	172 +26 298	505 +74 679	Renames a ten to the hundreds column when it is not necessary.
0	3	436 +11 547	172 +26 398	505 +74 1279	Adds correctly in the ones and tens column. The answer in the hundreds column is obtained by adding the digit in the hundreds column of the top addend to the digit in the tens column of the bottom addend. (e.g., 505 + 74 = 4 + 5 = 9; 7 + 0 = 7; 7 + 5 = 12; thus, answer is 1279.)
0	1	 436 +11 54.747	172 +26 39.538	505 +74 128.249	The mechanics of the short-form multiplication algorithm are used with the addition operation. The arrows in the problems indicate how the answers were derived. A decimal point is arbitrarily placed in the answer.
2	6				

Table 14

Systematic Errors in Addition Algorithm: Level 6 – Adding a Three-Digit + Two Digit Number; Renaming in the Ones Column Only

Number or Errors					
Normal	Handicapped				Error
13	1	476 +17 483	205 +86 281	754 +28 772	Does not rename the sum of the ones column.
4	2	476 +17 583	205 +86 381	754 +28 872	Renames the sum of the ones column to the hundreds column instead of the tens column.
3	4	476 +17 4813	205 +86 2811	754 +28 7712	The sum of the ones column is placed in the answer without renaming the ten.
2	0	476 +17 813	205 +86 811	754 +28 712	The sum of the ones column is placed in the answer without renaming the ten. The digit from the hundreds column is not recorded in the answer.
0	1	476 +17 48	205 +86 28	754 +28 77	Ignores addition in the ones column. Adds in the tens and hundreds column.
1	0	476 +17 417	205 +86 286	754 +28 728	No addition is performed. The bottom addend is copied for the answer along with the digit in the hundreds column in the top addend.
1	0	476 +17 25	205 +86 21	754 +28 26	Adds each digit separately disregarding columns. (476 + 17 = 4 + 7 + 6 + 1 + 7 = 25)
1	0	476 +17 488	205 +86 287	754 +28 779	The answer in the ones column is one more than the digit in the ones column of the second addend. Addition in tens and hundreds column is correct.

Table 14(continued)

1	0	476 +17 93	205 +86 91	754 +28 82	The digit from the hundreds column is not recorded in the answer.
1	0	476 +17 413	205 +86 211	754 +28 712	The sum of the ones column is placed in the answer without renaming the ten. No addition is performed in the tens column. The digit from the hundreds column is recorded in the answer.
$\frac{1}{28}$	$\frac{0}{8}$	476 +17 584	205 +86 382	754 +28 873	The sum of the ones column is one more than it should be. Renames the sum of the ones column to the hundreds column instead of the tens column.

Table 15

Systematic Errors in Addition Algorithm: Level 7 - Adding a Three-Digit + a Two-Digit Number; Renaming in the Ones and Tens Columns

Number of Errors		Error	
Normal	Handicapped		
2	1	519 345 483 +82 +76 +57 591 311 430	Does not rename the sum of the ones or tens column.
2	0	519 345 483 +82 +76 +57 437 269 426	Subtracts instead of adds.
1	3	519 345 483 +82 +76 +57 691 411 530	Did not rename the sum of the ones column. Renamed correctly the sum of the tens column.
1	0	519 345 483 +82 +76 +57 5911 31111 41310	The sum of the ones and tens columns are placed in the answer without renaming.
0	1	519 +82 8 (9 x 2 = 18) 9 (8x1=8;8+1=9) 5 (5 is brought down) 598 345 +76 0 (3 x 5 = 30) 31 (7 x 4 = 28; 28 + renamed 3 = 31) 310 483 +57 1 (7 x 3 = 21) 2 (5 x 8 = 40; 40 + renamed 2 = 42) 16 (4 x renamed 4 = 16) 1621	Multiplication is attempted. Each digit of the bottom addend is separately multiplied by one of the digits in the top addend. Various ways of renaming are attempted.

Table 15 (continued)

0	1	519 +82 791	345 +76 511	483 +57 630	Renames both the sum of the ones and tens column to the hundreds column.
$\frac{0}{6}$	$\frac{1}{7}$	01 519 +82 511	21 345 +76 511	30 483 +57 711	The wrong digit is renamed.

Table 16

Systematic Errors in Addition Algorithm:
Level 8 - Column Addition--Three Two-Digit Numbers; With Renaming

Number of Errors					
Normal	Handicapped	Error			
1	0	52 86 +14 142	19 27 +73 109	14 45 +61 110	Did not rename the ten from the sum of the ones column.
1	0	52 86 +14 26	19 27 +73 29	14 45 +61 21	Adds each digit separately disregarding ones and tens column. (52 + 86 + 14 = 2 + 6 + 4 + 5 + 8 + 1 = 26)
1	0	52 86 +14 153	19 27 +73 120	14 45 +61 121	Answer in the ones column is one more than it should be.
0	2	52 86 +14 52	19 27 +73 19	14 45 +61 20	Addition correct. Digit in the hundrends column is not recorded.
0	1	52 86 +14 1512	19 27 +73 1119	14 45 +61 1210	The sum of the ones column is placed in the answer. However, a ten is renamed to the tens column.
0	1	52 86 +14 1412	19 27 +73 1019	14 45 +61 1110	The sum of the ones column is placed in the answer without renaming the ten.
0	1	52 86 +14 42	19 27 +73 09	14 45 +61 10	Did not add the renamed digit from the ones column to the sum of the tens column. Did not record the hundreds digit of the sum of the tens column in the answer.
3	5				

Figure 1
Example of Data Sheets

Name: ______________________________ Level 4: Addition

Grade: ____________

Teacher's Name: ______________________

School: ____________________________

Date: ____________

24	56	17	25	32
+ 67	+ 28	+33	+ 16	+29

Report No. 3

Systematic Errors in the Subtraction Algorithm in Normal and Handicapped Populations[1]

Linda S. Cox

This paper is part of a continuing report on the descriptive research on systematic computational errors in the four algorithms. In an earlier paper (Cox, 1973) information regarding the selection, size and characteristics of the sample was reported in detail along with the data relative to the addition algorithm. This paper presents only the data relative to the subtraction algorithm. A brief summary of the research procedures is included here.

Systematic Errors Defined

An error is labeled a systematic error when there is a repeatedly occurring incorrect response that is evident in a specific algorithmic computation. This incorrect process will be evident in three out of five problems of a given type. Systematic errors must be distinguished from random errors. Random errors give no evidence of a reoccurring incorrect process of thinking or recording. This definition is similar to Grossnickle's (1935) definition of constant errors in long division.

Population

Normal and handicapped classrooms were chosen from the Shawnee Mission Public School system, a suburb of Kansas City. A private and a parochial school also participated. The total sample size was 744.

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[1] This study was supported by grant number NS 05362 from the National Institute of Neurological Diseases and Stroke to the Bureau of Child Research, University of Kansas.

Procedures

Six levels of skill (see Table 17) were identified in the subtraction algorithm. They were organized by the number of digits and the existence of renaming in the algorithm. Data sheets (see Figure 2) were distributed each week beginning in November, 1972, through February, 1973.

Insert Table 17 (p.53) here

Insert Fig. 2 (p.66) here

Classroom teachers administered the data sheets. Two requirements had to be met before a child's paper was included for analysis. Those requirements were:

1. The child had been taught the basic facts that pertain to the algorithm being studied. The child may not score 100% on a test of the facts, but he knows most of them.

2. The child has been exposed to the computational skill levels of the algorithm although he may not be proficient in its use.

If a child did not meet the above two requirements his paper was not included for analysis. If a child did meet these requirements his paper was analyzed and classified in only one of the following categories.

Systematic Error. The child missed at least three out of five problems recording repeatedly the same incorrect type of response in the algorithm.

Random Error. The child missed at least three out of five problems but a pattern or systematic incorrect process could not be detected.

Careless Error. The child missed only one or two problems out of five. He basically knows how to work the algorithm.

No Error. All five problems were correct.

Incomplete Data Sheet. The child did not work all five problems so that the data sheet could not be classified as one of the above types. Great effort was made to keep this category as small as possible during the data collection.

Results

Table 18 shows the frequency of all types of errors while Table 19 displays only the analysis of systematic errors. Tables 22 through 27 illustrate and define all of the systematic errors.

Insert Table 18 (p.54) here

Table 18 clearly reveals that more children make systematic errors than random errors (13% vs. 8%).

Table 19 shows the percentages of children making systematic errors in the subtraction algorithm. Children's papers were analyzed only if they made the same error in three out of five problems that they worked.

Insert Table 19 (p.55) here

Table 19 should be read as follows: For skill level 1 (subtracting a one-digit from a two-digit number with no renaming) 6% of the 96 papers that were analyzed in grade 2 normal classrooms were classified as systematic errors in the subtraction algorithm. This was also true for the 91 papers that were analyzed in grade 3 (6%), but the frequency dropped to 3% of the 110 papers in grade 4. Average number of papers analyzed per skill level is the same number as the average number of children per skill level who met the requirements for inclusion in the study. Overall, 5% of the total population who met the requirements for inclusion in the study made systematic errors on skill level 1.

Table 20 projects the number of estimated systematic errors for typical classrooms of 30 students in normal populations. Table 21 projects these ratios for handicapped classrooms.

Insert Tables 20 & 21 (p.56) here

Teachers at every grade level can anticipate having to deal with the problem of systematic errors in subtraction. Teachers of the handicapped and teachers in grades 2, 3, and 4 can expect a large number of systematic errors under the present methods of teaching the subtraction algorithm. That is to say, that unless methods of teaching the subtraction algorithm are changed, teachers can expect to encounter several types of systematic errors in subtraction each year. The ratios in Tables 20 and 21 should <u>not</u> be interpreted to mean that in Grade 2, 15 out of the 30 children will make systematic errors; but that 15 systematic errors will occur across the six skill levels for every 30 children in grade 2. It is quite likely that approximately 6% of the 30 children or one or two children per 30 will make those errors. It is most likely that these one or two children are accounting for the 15 systematic errors that are being made across the six different skill levels.

Tables 22 through 27 illustrate and define all of the systematic errors by population groups. Examples of the actual errors were taken from the data sheets.

Insert Tables 22-27 (pp.57-64) here

Discussion:

Over twice the number of students made systematic errors in subtraction than in addition. Table 19 shows that of the total population, 13% made systematic errors in subtraction. The earlier paper (Cox, 1973) reported that 5% of the students made systematic errors in addition. Also, twice as many systematic errors were made in subtraction when renaming was a part of the computational process. Skill levels 2, 4, 5, and 6 involved renaming. The percentages of error were greater in those levels than in skill levels 1 and 3.

Since the child matures in his thinking as he progresses through the grades, one would expect that the percentage of systematic errors would decline as a child's intellectual skills mature. This was the case for the normal population in the addition algorithm (Cox, 1973). However, in the subtraction algorithm an increase between 2nd and 3rd grade was noted with a subsequent decrease after 3rd grade. The percentages were:

	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
Subtraction:	13%	23%	8%	6%	6%
Addition:	10%	6%	1%	1%	0%

This same type of pattern was also evident in the handicapped population with 15% making systematic errors in primary classrooms, increasing to 24% in intermediate classrooms. One explanation for this pattern is that not all of the skill levels in the subtraction algorithm are introduced in the 2nd grade or primary handicapped classrooms. Third grade and intermediate handicapped classrooms fully develop the subtraction algorithm and this is where the highest frequency of systematic errors occur.

Two important points that were presented in the Results section should be reemphasized. One is that every teacher can expect to encounter at least one child each year who will make systematic errors in subtraction. The other point is that more children made systematic errors than random errors (13% vs 8%). This is important to note because action can be taken to correct these errors. They can be identified and once identified, a remedial instructional program developed and implemented. This is not the case with random errors. With random errors, the child makes many different errors so that they are much more difficult to treat. Fortunately, on the basis of this study, fewer children make random errors than systematic errors.

Many of the systematic errors in subtraction dealt with errors in renaming the minuend. In examining the descriptions of systematic errors in Tables 22 through 27, one finds that there are a total of 52 systematic errors. Of these 52, 29 or 56% were due to failures in renaming the minuend.

The other types of systematic errors dealt with the general concepts regarding the meaning of number, place value, and the operation of subtraction. For example, in the systematic error of the type 37 - 4 = 3, the child must not realize that 37 means 3 tens and 7 ones from which 4 ones are being subtracted. Arriving at an answer of 3 reveals that the child has no ability to judge if the answer is reasonable. Not knowing if the answer is reasonable indicates failure to understand the meaning of the number 37 and the meaning of the subtraction operation. Of the children making systematic errors on skill level 1 (Table 22), this error was made by 47% of the normal population and 13% of the handicapped population. It must be emphasized that all children had been exposed to the algorithm process that was needed to solve the problem. Errors like this with their subsequent high frequency of occurrence indicate that it is quite evident that serious attention must be given to diagnosing systematic errors. For the classroom teacher, a method of attack is to identify the systematic error and begin remedial instruction at that point. Early identification cannot be overemphasized.

It is not within the scope of this paper to present ideas regarding remedial instructional techniques. Presently, no clear didactic model has been shown to overcome these specific types of errors. This is a problem for future research. Meanwhile teachers will have to try a variety of methods. It is certain that a very big factor in eliminating these errors will be teacher perception of them because the remedial process begins with the teacher and his/her perception and recognition of the problem.

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Table 17

Levels of Skill in Subtraction[1]

Level 1:	Subtracting a one-digit from a two-digit number; no renaming.	29 -7
Level 2:	Subtracting a one-digit from a two-digit number; with renaming.	32 -8
Level 3:	Subtracting a two-digit from a two-digit number; no renaming.	46 -12
Level 4:	Subtracting a two-digit from a two-digit number; with renaming.	53 -14
Level 5:	Subtracting a two-digit from a three-digit number; renaming in ones column.	453 -45
Level 6:	Subtracting a three-digit from a four-digit number; with renaming.	4,602 -794

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[1] These levels are organized by the number of digits and the existence of renaming in the algorithm.

Table 18

All Types of Errors for Both Populations for All Grades in Subtraction Algorithm

	Systematic Errors	Random Errors	Careless Error	No Error	Incomplete Data Sheet
Skill Levels	3 out of 5	3 out of 5	1 or 2 out of 5		
1	5%	4%	15%	76%	0% = 100%
2	17%	11%	22%	49%	1% = 100%
3	2%	4%	13%	81%	0% = 100%
4	22%	8%	22%	46%	2% = 100%
5	10%	7%	22%	60%	1% = 100%
6	23%	11%	27%	38%	1% = 100%
Average	13%	8%	20%	58%	1% = 100%

Table 19

Percentage of Systematic Errors in Subtraction Algorithm by Population, Grade, and Skill Level

Levels of Skill	Normal Population Grades					Handicapped Population Classrooms			Both Populations
	2	3	4	5	6	Primary	Intermediate	Jr.Hi.	
1	6%	6%	3%	*	*	8%	7%	3%	5%
2	14%	23%	3%	*	*	20%	41%	9%	17%
3	2%	1%	1%	*	*	0%	5%	4%	2%
4	29%	28%	7%	*	*	33%	41%	14%	22%
5	*	14%	8%	4%	*	*	11%	20%	10%
6	*	64%	24%	8%	6%	*	37%	19%	23%
Total # of Papers Analyzed	382	544	657	199	100	33	296	340	2551
Average # of Papers Analyzed/ Skill Level	96	91	110	96	100	8	49	67	
Average % by Grade Level	13%	23%	8%	6%	6%	15%	24%	12%	13%

*Not tested at this grade level.

Table 20

Estimated Systematic Errors Across the Six Different Skill Levels for Typical Classrooms of 30 Students

Grade	Estimated Number of Errors
Grade 2	15
Grade 3	33
Grade 4	15
Grade 5	4
Grade 6	2

Table 21

Estimated Systematic Errors Across the Six Different Skill Levels for Handicapped Classrooms

Instructional Level	Estimated Number of Errors	Typical Size of Group
Primary	5	10
Intermediate	11	10
Junior High	10	16

Table 22

Systematic Errors in Subtraction Algorithm:

Level 1 - Subtracting a One-Digit from a Two-Digit Number;

No Renaming

Number of Errors Normal	 Handicapped	Error
7	1	$\begin{array}{r}37\\-4\\\hline 3\end{array}$ $\begin{array}{r}43\\-1\\\hline 2\end{array}$ $\begin{array}{r}85\\-3\\\hline 2\end{array}$ Subtracted in the ones column but ignored the tens column.
4	2	$\begin{array}{r}37\\-4\\\hline 41\end{array}$ $\begin{array}{r}43\\-1\\\hline 44\end{array}$ $\begin{array}{r}85\\-3\\\hline 88\end{array}$ Addition is performed.
1	3	$\begin{array}{r}37\\-4\\\hline 13\end{array}$ $\begin{array}{r}43\\-1\\\hline 32\end{array}$ $\begin{array}{r}85\\-3\\\hline 52\end{array}$ Subtracted the single digit of the subtrahend from both digits of the minuend.
2	1	$\begin{array}{r}37\\-4\\\hline 23\end{array}$ $\begin{array}{r}43\\-1\\\hline 32\end{array}$ $\begin{array}{r}85\\-3\\\hline 72\end{array}$ "Borrowed" from the tens column when it was unnecessary.
0	1	$\begin{array}{r}37\\-4\\\hline 43\end{array}$ $\begin{array}{r}43\\-1\\\hline 12\end{array}$ $\begin{array}{r}85\\-3\\\hline 32\end{array}$ Two explanations are possible: 1.) Subtracts correctly in the ones column and places the digit of the subtrahend in the tens column of the answer; or 2.) Subtracts correctly in the ones column. Subtracts the digits of the minuend to arrive at the answer in the tens column, e.g., 85 - 3 = 5 - 3 = 2 ones; 8 - 5 (the digits in the minuend) = 3 and this digit is placed in the tens column of the answer. Answer = 32.
1	0	$\begin{array}{r}\overset{13}{\cancel{3}}7\\-4\\\hline 93\end{array}$ $\begin{array}{r}\overset{14}{\cancel{4}}3\\-1\\\hline 132\end{array}$ $\begin{array}{r}\overset{18}{\cancel{8}}5\\-3\\\hline 152\end{array}$ Subtracted the single digit of the subtrahend from both digits of the minuend. Converted the tens column of the minuend into a two-digit number.
15	8	

Table 23

Systematic Errors in Subtraction Algorithm:

Level 2 - Subtracting a One-Digit from a Two-Digit Number;

With Renaming

Number of Errors					
Normal	Handicapped	Error			
27	16	32 -6 34	50 -8 58	24 -5 21	Did not regroup the minuend. Subtracted the larger subtrahend from the smaller minuend.
0	6	32 -6 36	50 -8 52	24 -5 29	Renamed the minuend and subtracted correctly in the ones column; but recorded the original ten of the minuend in the answer.
3	2	32 -6 6	50 -8 2	24 -5 9	Renamed and subtracted correctly in the ones column but failed to record the ten in the tens column.
2	1	32 -6 04	50 -8 08	24 -5 01	Did not regroup the minuend. Subtracted the smaller minuend from larger subtrahend in the ones column. Wrote a zero in the tens column.
1	1	32 -6 30	50 -8 50	24 -5 20	Placed a zero in the ones column and brought down the digit in the tens column and placed it in the answer.
2	0	32 -6 38	50 -8 58	24 -5 29	Addition is performed.
2	0	$\overset{2}{\cancel{3}}2$ -6 24	$\overset{4}{\cancel{5}}0$ -8 48	$\overset{1}{\cancel{2}}4$ -5 11	Regrouped the minuend but then ignored the regrouping and subtracted the smaller minuend from the larger subtrahend.

Table 23 (continued)

2	0	$\begin{array}{r}32\\-6\\\hline 34\end{array}$ $\begin{array}{r}50\\-8\\\hline 48\end{array}$ $\begin{array}{r}24\\-5\\\hline 31\end{array}$ Did not regroup the minuend. Subtracted the smaller minuend from the larger subtrahend. The digit in the tens column in the answer is the same, one larger, or one smaller than the digit in the tens column of the minuend.
0	1	$\begin{array}{r}32\\-6\\\hline 48\end{array}$ $\begin{array}{r}50\\-8\\\hline 68\end{array}$ $\begin{array}{r}24\\-5\\\hline 39\end{array}$ Addition is performed in the ones column. An extra ten is added to the ten column and then addition is performed in that column.
0	1	$\begin{array}{r}32\\-6\\\hline 32\end{array}$ $\begin{array}{r}50\\-8\\\hline 50\end{array}$ $\begin{array}{r}24\\-5\\\hline 24\end{array}$ No subraction is performed. The answer is either the minuend or the subtrahend.
1	0	$\begin{array}{r}32\\-6\\\hline 16\end{array}$ $\begin{array}{r}50\\-8\\\hline 32\end{array}$ $\begin{array}{r}24\\-5\\\hline 9\end{array}$ Correctly renamed the minuend. Correctly subtracted in the ones column. Subtracted an extra ten in the tens column so that the answer is always ten less than it should be.
1	0	$\begin{array}{r}32\\-6\\\hline 27\end{array}$ $\begin{array}{r}50\\-8\\\hline 49\end{array}$ $\begin{array}{r}24\\-5\\\hline 16\end{array}$ In the ones column the answer is one more than the digit in the subtrahend. The answer in the tens column is one less than the ten in the minuend.
1	0	$\begin{array}{r}32\\-6\\\hline 5\end{array}$ $\begin{array}{r}50\\-8\\\hline 7\end{array}$ $\begin{array}{r}24\\-5\\\hline 4\end{array}$ No subtraction is performed. The number in the answer is one less than the subtrahend.
1	0	$\begin{array}{r}32\\-6\\\hline 26\end{array}$ $\begin{array}{r}50\\-8\\\hline 48\end{array}$ $\begin{array}{r}24\\-5\\\hline 15\end{array}$ No subtraction is performed. The answer in the ones column is the same as the subtrahend. The answer in the tens column is one less than the minuend.
43	28	

Table 24

Systematic Errors in Subtraction Algorithm:

Level 3 - Subtracting a Two-Digit from a Two-Digit Number;

No Renaming

Number of Errors		Error	
Normal	Handicapped		
1	3	$\begin{array}{r}49\\-11\\\hline 60\end{array}\quad\begin{array}{r}28\\-16\\\hline 44\end{array}\quad\begin{array}{r}83\\-32\\\hline 115\end{array}$	Addition is performed.
3	0	$\begin{array}{r}\overset{3\,19}{\cancel{4}\cancel{9}}\\-11\\\hline 218\end{array}\quad\begin{array}{r}\overset{1\,18}{\cancel{2}\cancel{8}}\\-16\\\hline 12\end{array}\quad\begin{array}{r}\overset{7\,13}{\cancel{8}\cancel{3}}\\-32\\\hline 411\end{array}$	Renamed the minuend when it was unnecessary. The difference in the ones column is a two digit number. The two-digit number is placed in the answer.
$\frac{0}{4}$	$\frac{1}{4}$	$\begin{array}{r}49\\-11\\\hline 49\end{array}\quad\begin{array}{r}28\\-16\\\hline 28\end{array}\quad\begin{array}{r}83\\-32\\\hline 32\end{array}$	Used either the minuend or the subtrahend for the answer.

Table 25

Systematic Errors in Subtraction Algorithm:

Level 4 - Subtracting a Two-Digit from a Two-Digit Number;

With Renaming

Number of Errors		Error
Normal	Handicapped	
48	23	$\begin{array}{r}53\\-14\\\hline 41\end{array}$ $\begin{array}{r}72\\-56\\\hline 24\end{array}$ $\begin{array}{r}45\\-19\\\hline 34\end{array}$ Did not rename. Subtracted smaller minuend from larger subtrahend in the ones column.
3	3	$\begin{array}{r}53\\-14\\\hline 49\end{array}$ $\begin{array}{r}72\\-56\\\hline 26\end{array}$ $\begin{array}{r}45\\-19\\\hline 36\end{array}$ Renamed the minuend and subtracted correctly in the ones column. Subtracted in the tens column without considering that the minuend had been renamed.
4	0	$\begin{array}{r}53\\-14\\\hline 49\end{array}$ $\begin{array}{r}72\\-56\\\hline 66\end{array}$ $\begin{array}{r}45\\-19\\\hline 36\end{array}$ Renamed the minuend and correctly subtracted in the ones column. No subtraction is performed in the tens column. The renamed ten in in the minuend is placed in the answer.
1	0	$\begin{array}{r}53\\-14\\\hline 43\end{array}$ $\begin{array}{r}72\\-56\\\hline 22\end{array}$ $\begin{array}{r}45\\-19\\\hline 35\end{array}$ No subtraction is performed in the ones column. The answer in the ones column is identical to the ones digit in the minuend. Subtraction in the tens column is performed without renaming the minuend.
1	0	$\begin{array}{r}5\overset{10}{\cancel{3}}\\-14\\\hline 36\end{array}$ $\begin{array}{r}7\overset{10}{\cancel{2}}\\-56\\\hline 14\end{array}$ $\begin{array}{r}4\overset{10}{\cancel{5}}\\-19\\\hline 21\end{array}$ Incorrectly renamed the minuend by placing a ten in the ones column. Subtracted correctly in the tens column.
1	0	$\begin{array}{r}53\\-14\\\hline 40\end{array}$ $\begin{array}{r}72\\-56\\\hline 20\end{array}$ $\begin{array}{r}45\\-19\\\hline 30\end{array}$ No subtraction is performed in the ones column. A zero is placed in ones column. Subtraction in the tens column is performed without renaming the minuend.
58	26	

Table 26

Systematic Errors in Subtraction Algorithm:

Level 5 - Subtracting a Two-Digit from a Three-Digit Number:

Renaming in Ones Column

Number of Errors					
Nomal	Handicapped	Error			
11	7	$\begin{array}{r}493\\-45\\\hline 452\end{array}$	$\begin{array}{r}376\\-58\\\hline 322\end{array}$	$\begin{array}{r}830\\-17\\\hline 827\end{array}$	Did not rename the minuend. In the ones column subtracted the smaller minuend from the larger subtrahend.
2	2	$\begin{array}{r}493\\-45\\\hline 358\end{array}$	$\begin{array}{r}376\\-58\\\hline 228\end{array}$	$\begin{array}{r}830\\-17\\\hline 723\end{array}$	Renamed the hundreds column instead of the tens column.
2	1	$\begin{array}{r}493\\-45\\\hline 048\end{array}$	$\begin{array}{r}376\\-58\\\hline 018\end{array}$	$\begin{array}{r}830\\-17\\\hline 013\end{array}$	A zero is placed in the hundreds column in the answer.
2	1	$\begin{array}{r}493\\-45\\\hline 458\end{array}$	$\begin{array}{r}376\\-58\\\hline 328\end{array}$	$\begin{array}{r}830\\-17\\\hline 823\end{array}$	Renamed the minuend and subtracted correctly in the ones column. Subtracted in the tens column without considering that the minuend had been renamed.
0	2	$\begin{array}{r}493\\-45\\\hline 538\end{array}$	$\begin{array}{r}376\\-58\\\hline 434\end{array}$	$\begin{array}{r}830\\-17\\\hline 847\end{array}$	Addition is performed.
3	0	$\begin{array}{r}493\\-45\\\hline 348\end{array}$	$\begin{array}{r}376\\-58\\\hline 218\end{array}$	$\begin{array}{r}830\\-17\\\hline 713\end{array}$	Renamed hundreds column when it was unnecessary.
1	1	$\begin{array}{r}493\\-45\\\hline 52\end{array}$	$\begin{array}{r}376\\-58\\\hline 22\end{array}$	$\begin{array}{r}830\\-17\\\hline 27\end{array}$	Did not rename. Subtracted smaller minuend from larger subtrahend. Did not record the digit in the hundreds column in the answer.

Table 26 (cont.)

2	0	$\begin{array}{r} 493 \\ -45 \\ \hline 450 \end{array}$ $\begin{array}{r} 376 \\ -58 \\ \hline 320 \end{array}$ $\begin{array}{r} 830 \\ -17 \\ \hline 820 \end{array}$	No subtraction is performed in the ones column. A zero is placed in the ones column for the answer. Subtraction in the tens column is performed without renaming the minuend.
0	1	$\begin{array}{r} 4\overset{12}{\cancel{9}}\overset{13}{\cancel{3}} \\ -45 \\ \hline 388 \end{array}$ $\begin{array}{r} 3\overset{15}{\cancel{7}}\overset{16}{\cancel{6}} \\ -58 \\ \hline 308 \end{array}$ $\begin{array}{r} 8\overset{9}{\cancel{3}}\overset{10}{\cancel{0}} \\ -17 \\ \hline 783 \end{array}$	Incorrectly renamed the minuend. Renamed number in tens column is one less than renamed number in the ones column.
1	0	$\begin{array}{r} 49\overset{10}{\cancel{3}} \\ -45 \\ \hline 445 \end{array}$ $\begin{array}{r} 37\overset{10}{\cancel{6}} \\ -58 \\ \hline 312 \end{array}$ $\begin{array}{r} 83\overset{10}{\cancel{0}} \\ -17 \\ \hline 813 \end{array}$	Incorrectly renamed the minuend by placing a ten in the ones column. Correctly subtracted in the tens column.
1	0	$\begin{array}{r} 4\overset{5}{\cancel{9}}\overset{13}{\cancel{3}} \\ -45 \\ \hline 418 \end{array}$ $\begin{array}{r} 3\overset{2}{\cancel{7}}\overset{16}{\cancel{6}} \\ -58 \\ \hline 338 \end{array}$ $\begin{array}{r} 2\overset{3}{\cancel{8}}\overset{15}{\cancel{5}} \\ -39 \\ \hline 206 \end{array}$	Incorrectly renamed the minuend. The renamed number in the tens column is obtained by subtracting the smaller digit from the larger digit in the tens column. Subtraction in the tens column is then performed with this renamed number; e.g., the renamed ten of 493 - 45 is obtained by subtracting 9 - 4 = 5. Five is the renamed number.
25	15		

Table 27

Systematic Errors in Subtraction Algorithm:

Level 6 - Subtracting a Three-Digit from a Four-Digit Number; With Renaming

Number of Errors			
Normal	Handicapped	Error	
40	13	$\begin{array}{r} \overset{3\,16\,9\,12}{\cancel{4}\cancel{6}\cancel{0}\cancel{2}} \\ -794 \\ \hline 3908 \end{array}$ $\begin{array}{r} \overset{2\,10\,14\,16}{\cancel{3}\cancel{1}\cancel{4}\cancel{6}} \\ -589 \\ \hline 2567 \end{array}$ $\begin{array}{r} \overset{1\,13\,11\,17}{\cancel{2}\cancel{3}\cancel{1}\cancel{7}} \\ -948 \\ \hline 1479 \end{array}$	Multiple errors in renaming the minuend.
13	7	$\begin{array}{r} 4602 \\ -794 \\ \hline 4192 \end{array}$ $\begin{array}{r} 3146 \\ -589 \\ \hline 3443 \end{array}$ $\begin{array}{r} 2317 \\ -948 \\ \hline 2631 \end{array}$	Did not rename the minuend. Subtracted the smaller minuend from the larger subtrahend.
3	3	$\begin{array}{r} 3146 \\ -589 \\ \hline 3667 \end{array}$ $\begin{array}{r} 5021 \\ -654 \\ \hline 5477 \end{array}$ $\begin{array}{r} 3104 \\ -287 \\ \hline 3927 \end{array}$	Did not allow for having renamed in the tens, hundreds, or thousands column.
7	1	$\begin{array}{r} 4602 \\ -794 \\ \hline 3818 \end{array}$ $\begin{array}{r} 5021 \\ -654 \\ \hline 4467 \end{array}$ $\begin{array}{r} 3104 \\ -287 \\ \hline 2827 \end{array}$	Incorrectly renamed when a zero necessitated renaming twice.
5	0	$\begin{array}{r} 4602 \\ -794 \\ \hline 3898 \end{array}$ $\begin{array}{r} 2317 \\ -948 \\ \hline 1349 \end{array}$ $\begin{array}{r} 5021 \\ -654 \\ \hline 4667 \end{array}$	When a zero or a number renamed as ten is in the minuend, the digit in the subtrahend is placed in the answer.
0	1	$\begin{array}{r} 4\overset{10\,11\,12}{\cancel{6}\cancel{0}\cancel{2}} \\ -794 \\ \hline 4328 \end{array}$ $\begin{array}{r} 3\overset{14\,15\,16}{\cancel{1}\cancel{4}\cancel{6}} \\ -589 \\ \hline 3977 \end{array}$ $\begin{array}{r} 3\overset{12\,13\,14}{\cancel{1}\cancel{0}\cancel{4}} \\ -287 \\ \hline 3057 \end{array}$	Renames each digit in the minuend as one less than the digit on the right was renamed.
2	0	$\begin{array}{r} 4\overset{99}{\cancel{6}\cancel{0}}2 \\ -794 \\ \hline 3208 \end{array}$ $\begin{array}{r} 3\overset{99}{\cancel{1}\cancel{4}}6 \\ -589 \\ \hline 2417 \end{array}$ $\begin{array}{r} 3\overset{99}{\cancel{1}\cancel{0}}4 \\ -287 \\ \hline 2717 \end{array}$	The tens and hundreds column are renamed as 9.

Table 27 (cont.)

1	0	$4\overset{7}{\cancel{6}}02$ -794 3092	$3\overset{2}{\cancel{1}}46$ -589 2343	$231\overset{8}{\cancel{7}}$ -948 2600	Renamed the minuend incorrectly by adding a "one" to one of the numbers. Subtracted smaller minuend from larger subtrahend.
1	0	4602 -794 5396	3146 -589 3735	2317 -948 3265	Addition is performed.
1	0	4602 -794 3908	2317 -948 1469	5021 -654 4467	Did not rename in the hundreds column. Renamed correctly in the other columns.
1	0	$\overset{3}{\cancel{4}}\overset{16}{\cancel{6}}0\overset{12}{\cancel{2}}$ -794 3908	$5\overset{7}{0}\overset{11}{\cancel{2}}\overset{11}{\cancel{1}}$ -654 5367	$\overset{2}{\cancel{3}}\overset{11}{\cancel{1}}0\overset{14}{\cancel{4}}$ -287 2917	When a zero is in the minuend, number on its left is not renamed.
1	0	$46\overset{11}{\cancel{0}}2$ -794 3829	$5\overset{11}{\cancel{0}}21$ -654 4567	$31\overset{11}{\cancel{0}}4$ -287 2837	The zero is the minuend is renamed as eleven.
75	25				

Figure 2
Example of Data Sheets

Level 4: Subtraction

Name: ____________________

Grade: ____________

Teacher's Name: ____________________

School: ____________________

Date: ____________

53	72	45	87	60
-14	- 56	-19	-78	-13

Report No. 4

Systematic Errors in the Multiplication Algorithm in Normal and Handicapped Populations[1]

Linda S. Cox

This paper is the third in a series of reports from a larger research study to classify the frequency and type of errors that are made by normal and handicapped children. These errors are in the whole number operations in arithmetic. Results concerning the addition and subtraction algorithms have been reported in earlier papers (Cox, 1973 a, b). The current report deals with the data on the multiplication algorithm. Only a summary of the procedures and a brief summary of the literature is presented here since more detailed information regarding the entire research study was presented in the first paper on the addition algorithm.

Literature

Persistent errors in various arithmetical computations were first recognized by Myers (1924). The reliability of the diagnosis of persistent errors was established by Brueckner and Elwell (1932) and confirmed again by Bruecker (1935) in computational processes in rational numbers. Grossnickle (1935) reported another similar study but in the computational process in division of whole numbers.

Two recent research reports studied the overall nature of computational errors. Roberts (1972), in studying a third grade population, reported all types of computational errors in four categories. The categories with the percentage of occurrence for that error were wrong operation (18%), obvious computational error (18%),

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[1] This study was supported by grant number NS 05362 from the National Institute of Neurological Diseases and Stroke to the Bureau of Child Research, University of Kansas.

defective algorithm (36%), and random response (28%). The defective algorithm was the most frequent type within his four categories. This present study further analyzes the defective algorithms.

The results of another study (Ellis, 1972) support the general conclusion of other research in this area. By analyzing the written whole number computations of sixth grade students, Ellis concluded that it was profitable to analyze errors as a means of gathering data and planning individualized instruction. He also noted that a substantial number of errors of undetermined origin emphasized a need for a more thorough analysis.

Some recent writers have stressed the importance of identifying all types of computational errors within the framework of diagnostic teaching of arithmetic. Literature with this emphasis includes Glennon and Wilson (1972), Ashlock (1972), and Reisman (1972).

There have not been any studies which have specifically analyzed the systematic errors in the multiplication algorithm by skill levels, grade levels, and by normal and handicapped populations.

Systematic Errors Defined

An error is labeled a systematic error when there is a repeatedly occuring incorrect response that is evident in a specific algorithmic computation. This incorrect process is apparent in three out of five problems of a given type. Systematic errors must be distinguished from random errors. Random errors give no evidence of a reoccurring incorrect process of thinking or recording. This definition is similar to Grossnickle's (1935) definition of constant errors in long division.

Population

Normal and handicapped classrooms were chosen from the Shawnee Mission Public School system, a suburb of Kansas City. The handicapped population consisted of Learning Problem classrooms and Educable Mentally Retarded classrooms in the Shawnee Mission Public Schools, and classrooms for the emotionally disturbed at the University of Kansas Medical Center. A private and a parochial school also particiapted. The total sample size was 744. Shawnee Mission is a middle-income suburb. Descriptive information regarding the population is included in the earlier paper on the addition algorithm (Cox, 1973a).

Procedures

Ten levels of skill (see Table 28) were identified in the multiplication algorithm. It should be noted that the levels of skill were not arranged in increasing order of difficulty. They were organized by the number of multipliers and the inclusion or exclusion of renaming. Data sheets (see Fig. 3) were distributed each week beginning in March, 1973, through May, 1973.

Insert Table 28 & Fig. 3 (pp. 76 & 96) here

Classroom teachers administered the data sheets. Two requirements had to be met before a child's paper was included for analysis. Those requirements were:

1. The child had been taught the basic facts that pertain to the algorithm being studied. The child may not score 100% on a test of the facts, but he knows most of them.

2. The child has been exposed to the computational skill levels of the algorithm although he may not be proficient in its use.

If a child did not meet the above two requirements his paper was not included for analysis. If a child did meet these requirements his paper was analyzed and classified in only one of the following categories.

Systematic error. The child missed at least three out of five problems, recording repeatedly the same incorrect type of response in the algorithm.

Random error. The child missed at least three out of five problems but a pattern or systematic incorrect process could not be detected.

Careless error. The child missed only one or two problems out of five. He basically knows how to work the algorithm.

No error. All five problems are correct.

Incomplete data sheet. The child did not work all five problems so that the data sheet could not be classified as one of the above types. Great effort was made to keep this category as small as possible during the data collection.

Results

Table 29 shows the frequency of all types of errors while Table 30 displays only the analysis of systematic errors. Percentages in Table 29 refer to children whose papers were classified as containing specific systematic errors.

Insert Table 29 (p.77) here

Table 29 shows that 6% of the papers that were analyzed were classified as containing a specific systematic error in multiplication computations. Skill levels 5, 8, and 10 show the highest percentages of systematic errors (11%, 11%, and 12%, respectively). This is two to four times greater than the occurrences at the other skill levels. Skill levels 5, 8, and 10 deal with zero in the medial position in one of the factors.

The number of papers that were analyzed decreased as the skill level increased. This occurred because the younger children could not meet the assumptions as outlined in the procedures as more difficult levels of skills were administered.

Insert Table 30 (p. 78) here

Table 30 shows the data for all systematic errors in multiplication. Table 30 should be read as follows: for skill level 1 (see Table 28), 9% of the 56 papers that were analyzed in third grade, normal classrooms were classified as a specific systematic error. Four percent of the 73 papers in fourth grade were so classified. The figures from the handicapped population showed 6% of the 21 intermediate papers and 4% of the 29 junior high papers were so classified. For both populations the average percentages for all grade levels was 6%.

Additional information in Table 30 indicates the number of papers analyzed per grade. This figure increases for the higher grade levels because more children met the requirements for the study.

The average percentage by grade level increases from third to fourth grade with a subsequent drop in fifth and sixth grade. A similar pattern appears in the handicapped population.

Tables 31 through 40 illustrate and define all of the systematic errors. The actual number of cases of each error is indicated along with the percentage of the total errors that it represents.

Insert Tables 31-40 (pp.79-94) here

Discussion

The following points should be emphasized.

1. The over-all percentage of systematic errors (see Table 30) for multiplication is in close agreement with the percentage reported for the addition algorithm (Cox, 1973a). For the addition algorithm, 5% was reported compared to 6% for the multiplication algorithm. The data for the subtraction algorithm revealed 13% (Cox, 1973b). This indicates that the basic rate of systematic errors may be rather stable (5-6%) with the subtraction algorithm representing a much more difficult algorithmic computation (13%). At the present time, the data for the division algorithm has not been completely analyzed so no conclusions regarding different systematic errors across the four algorithms can be made. Only a trend is indicated.

2. Multiple errors presented a special problem. This was particularly true for multiplication. In the addition and subtraction algorithm, the reported systematic error was usually the only error that was made. However, occasionally a child also made a careless error concomitant with a systematic error. For example, if in the addition problems (Cox, 1973a) the child errored in an addition fact and still made a systematic error in the mechanics of the problem, then it was counted as a systematic error.

In the multiplication problems, many more careless errors were made. If a careless error and a systematic error both occurred, only the systematic error was counted.

3. The occurrence of a systematic error in multiplication may indicate that the child knows quite a bit about arithmetic. Depending upon the error, it may

indicate that he knows the addition facts, multiplication facts, concept of multiplication, place value, renaming, the mechanics of "carrying," and the mechanics of placing the partial products. If he didn't know many of those concepts his errors would be so numerous and various that they couldn't be counted as systematic errors. In such cases the errors were counted as random errors. This factor accounts for the increase in the percentage of random errors in the multiplication algorithm as compared to the percentage of random errors in subtraction or addition (10%, 8%, 4%, respectively).

4. Two to four times more systematic errors occurred in skill levels 5, 8, and 10 than in the other skill levels. These levels are shown in Tables 35, 38, and 40. Special emphasis should be given to diagnosing these types of problems since children make many different types of systematic errors in these problems. Those skill levels all contain a zero in the medial position in one of the factors.

5. It is evident from Tables 31 through 40 that in many cases, only one child was found that made a particular systematic error. Even though only one child is making a systematic error, it is important for the teacher to be able to recognize and identify it.

Implications

The argument could be advanced that the teacher should just have the child work the problem aloud, verbalizing all of his steps. This requires a lot of the teacher's classroom time and requires that the child is fairly verbal. Just observing him work may reveal some systematic errors. However, a child may make different errors while he's being observed than the errors that he makes when he works alone. For example, he may look to the teacher for signs of approval, indicating that he is working the problem correctly.

Many errors can be detected using this analytic method. As one becomes more acquainted with the analysis and type of errors that are made, one becomes more confident and skillful in detecting systematic errors. This in turn increases one's ability to detect new systematic errors.

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Table 28

Levels of Skill in Multiplication*

Level 1:	One-digit multiplier; two digit multiplicand; no renaming.	34 x2
Level 2:	One-digit multiplier; two and three-digit multiplicand containing intermediate and terminal zeros; no renaming.	401 x4
Level 3:	One-digit multiplier; two-digit multiplicand; renaming to the tens column.	47 x2
Level 4:	One-digit multiplier; three-digit multiplicand; renaming to the tens and/or hundreds column.	216 x5
Level 5:	One-digit multiplier; three and four-digit multiplicand; renaming to zeros.	805 x4
Level 6:	Two-digit multipliers with zero in ones column; two and three-digit multiplicands; with and without renaming.	18 x10
Level 7:	Two-digit multipliers; three-digit multiplicands with no zeros in either factor; multiple renaming.	346 x28
Level 8:	Two digit multipliers; three-digit multiplicands with zeros in the medial digit of the multiplicand; multiple renaming.	507 x32
Level 9:	Three-digit multipliers; three-digit multiplicands with no zeros in either factor; multiple renaming.	456 x251
Level 10:	Three-digit multipliers with zeros in the medial digit of the multiplier; three-digit multiplicand with no zeros; multiple renaming.	457 x305

*These levels are not in order of increasing difficulty. They are organized by the number of multipliers and the inclusion or exclusion of renaming.

Table 29

All Types of Errors for Both Populations for All Grades in the Multiplication Algorithm

		Systematic Errors	Random Errors	Careless Error	No Error	Incomplete Data Sheet
Skill Levels	# of Papers Analyzed	3 out of 5	3 out of 5	1 or 2 out of 5		
1	456	3%	1%	11%	85%	0% = 100%
2	475	4%	2%	20%	72%	2% = 100%
3	419	3%	2%	21%	72%	2% = 100%
4	398	3%	11%	34%	48%	4% = 100%
5	360	11%	6%	22%	58%	3% = 100%
6	352	7%	7%	25%	59%	2% = 100%
7	238	3%	11%	50%	31%	5% = 100%
8	239	11%	10%	32%	44%	3% = 100%
9	185	5%	34%	45%	11%	5% = 100%
10	143	12%	16%	45%	22%	5% = 100%
Average		6%	10%	31%	50%	3% = 100%

Table 30

Percentage of Systematic Errors in Multiplication Algorithm by Population, Grade, and Skill Level

Levels of Skill	Normal Population Grades					Handicapped Population Classroom			Both Populations
	2	3	4	5	6	Primary	Intermediate	Jr.Hi.	
1	*	9%	4%	0%	0%	*	6%	4%	3%
2	*	8%	2%	1%	0%	*	14%	10%	4%
3	*	3%	5%	0%	0%	*	15%	5%	3%
4	*	12%	6%	0%	0%	*	3%	6%	3%
5	*	28%	11%	5%	2%	*	29%	37%	11%
6	*	0%	10%	5%	3%	*	32%	11%	7%
7	*	0%	4%	10%	0%	*	22%	10%	3%
8	*	0%	17%	6%	4%	*	33%	20%	11%
9	*	0%	6%	5%	0%	*	75%	0%	6%
10	*	0%	10%	15%	3%	*	38%	10%	12%
Average % by Grade Level		6%	8%	5%	1%		27%	11%	6%
Total # of Papers Analyzed/Grade		278	725	872	847		206	287	3,229
Average # of Papers Analyzed/Skill Level		56	73	87	85		21	29	

*Classrooms were not tested becasue no child could meet the assumptions for the study.

Table 31

Systematic Errors in Multiplication Algorithm Level 1: One-digit Multiplier; Two-digit Multiplicand; No Renaming

Normal		Handicapped			
Percentage	# of Errors	Percentage	# of Errors	Error	
90%	9	100%	4	$\begin{array}{r}43\\ x2\\ \hline 46\end{array}$ $\begin{array}{r}24\\ x1\\ \hline 24\end{array}$ $\begin{array}{r}313\\ x3\\ \hline 319\end{array}$	No multiplication is performed in the ten and the hundreds column. The multiplicand in the tens and hundreds column is placed in the product. Multiplied correctly in the ones column.
10%	1	0%	0	$\begin{array}{r}43\\ x2\\ \hline 45\end{array}$ $\begin{array}{r}24\\ x1\\ \hline 25\end{array}$ $\begin{array}{r}313\\ x3\\ \hline 316\end{array}$	Added instead of multiplying.
100%	10	100%	4		

Table 32

Systematic Errors in Multiplication Algorithm
Level 2: One-digit Multiplier; Two and Three-digit Multiplicand Containing Medial and Terminal Zeros; No Renaming

Normal		Handicapped		
Percentage	# of Errors	Percentage	# of Errors	Error
50%	5	23%	2	$\begin{array}{r}30\\ \text{x}6\\ \hline 30\end{array}\quad\begin{array}{r}200\\ \text{x}5\\ \hline 200\end{array}\quad\begin{array}{r}60\\ \text{x}2\\ \hline 60\end{array}$ No multiplication is performed. The multiplicand is placed in the answer.
20%	2	47%	4	$\begin{array}{r}30\\ \text{x}6\\ \hline 186\end{array}\quad\begin{array}{r}200\\ \text{x}5\\ \hline 1055\end{array}\quad\begin{array}{r}401\\ \text{x}4\\ \hline 444\end{array}$ Incorrectly multiplies by zero performing $n \times 0 = n$. (Note: Occasionally a careless error is also made as illustrated in the last example.
20%	2	10%	1	$\begin{array}{r}30\\ \text{x}6\\ \hline 36\end{array}\quad\begin{array}{r}200\\ \text{x}5\\ \hline 205\end{array}\quad\begin{array}{r}401\\ \text{x}4\\ \hline 405\end{array}$ Added instead of multiplying.
10%	1	0%	0	$\begin{array}{r}30\\ \text{x}6\\ \hline 1018\end{array}\quad\begin{array}{r}200\\ \text{x}5\\ \hline 1010\end{array}\quad\begin{array}{r}401\\ \text{x}4\\ \hline 1016\end{array}$ The ones and tens column of the product is derived by multiplying the multiplier by the left digit of the multiplicand. A ten is then written to the left of this number.
0%	0	10%	1	$\begin{array}{r}1\\ 30\\ \text{x}6\\ \hline 190\end{array}\quad\begin{array}{r}1\\ 200\\ \text{x}5\\ \hline 1100\end{array}\quad\begin{array}{r}1\\ 401\\ \text{x}4\\ \hline 1704\end{array}$ Renamed when it was unnecessary.
0%	0	10%	1	$\begin{array}{r}1\\ 30\\ \text{x}6\\ \hline 80\end{array}\quad\begin{array}{r}1\\ 60\\ \text{x}2\\ \hline 20\end{array}\quad\begin{array}{r}1\\ 401\\ \text{x}4\\ \hline 604\end{array}$ When a multiplication produces a two-digit number, the digit is renamed by placing only one of the digits in the product and placing the other digit on top of the multiplicand as a renamed number.
100%	10	100%	9	

Table 33

Systematic Errors in Multiplication Algorithm Level 3: One-digit Multiplier; Two-digit Multiplicand; Renaming to the Tens Column

Normal		Handicapped		
Percentage	# of Errors	Percentage	# of Errors	Error
29%	2	44%	3	$\begin{array}{r}1\\47\\\text{x2}\\\hline 44\end{array}\quad\begin{array}{r}2\\16\\\text{x4}\\\hline 24\end{array}\quad\begin{array}{r}2\\29\\\text{x3}\\\hline 47\end{array}$ Multiplication is performed in the ones column. The "carried" number is multiplied by the tens digit of the multiplicand and this product is placed in the tens column of the answer.
43%	3	14%	1	$\begin{array}{r}1\\47\\\text{x2}\\\hline 104\end{array}\quad\begin{array}{r}2\\16\\\text{x4}\\\hline 124\end{array}\quad\begin{array}{r}2\\29\\\text{x3}\\\hline 127\end{array}$ Added the "carried" digit before multiplying; e.g., 16 x 4 = 6 x 4 = 24; the renamed 2 is added to the 1 ten yielding a sum of 3 tens. 3 tens times 4 ones equals 12 tens. Thus, 124 is the answer.
0%	0	14%	1	$\begin{array}{r}47\\\text{x2}\\\hline 14\end{array}\quad\begin{array}{r}16\\\text{x4}\\\hline 24\end{array}\quad\begin{array}{r}29\\\text{x3}\\\hline 27\end{array}$ Placed the entire product of the ones column in the answer without renaming. Did not multiply in the tens column.
14%	1	0%	0	$\begin{array}{r}16\\\text{x4}\\\hline 14\end{array}\quad\begin{array}{r}29\\\text{x3}\\\hline 27\end{array}\quad\begin{array}{r}13\\\text{x5}\\\hline 11\end{array}$ No multiplication is performed. The product is two less than the top factor.

Table 33 (cont.)

14%	1	0%	0	16 x4 124	29 x3 227	13 x5 115	Did not rename and placed the entire product of the ones column in the answer. No multiplication is performed in the tens column. The number in the tens column of the top factor is placed in the product.
0%	0	14%	1	47 x2 121	13 x5 101	38 x2 121	Reversal: "Carried" the wrong number when renaming the product of ones column; e.g., in 47 x 2, 7 x 2 = 14. The 1 was written in the ones column and the 4 was "carried" to the tens column.
0% 100%	0 7	14% 100%	1 7	47 x2 84	16 x4 84	29 x3 127	The renamed digit is multiplied instead of added to the product; example: 16 x4 4 (6 x 4 = 24) 8 (4 x 1 = 4; 84 4 x renamed 2 = 8).

Table 34

Systematic Errors in Multiplication Algorithm
Level 4: One-digit Multiplier; Three-digit Multiplicand; Renaming to the Tens and/or Hundreds Column

Normal		Handicapped		
Percentage	# of Errors	Percentage	# of Errors	Error
12%	1	67%	2	$\begin{array}{r}842\\ \text{x}7\\ \hline 5644\end{array}$ $\begin{array}{r}758\\ \text{x}8\\ \hline 2104\end{array}$ $\begin{array}{r}376\\ \text{x}5\\ \hline 610\end{array}$ After multiplying correctly in the ones column, multiplication is performed between the "carried" number and the multiplicand. If there is no "carried" number, the multiplication is performed in the usual manner between the multiplier and the multiplicand.
38%	3	0%	0	$\begin{array}{rl}539 & \\ \text{x}3 & \\ \hline 7 & (9\text{x}3=27)\\ 5 & (2+3=5;\ 5\text{x}3=15)\\ 18 & (1+5=6;\ 6\text{x}3=18)\\ \hline 1857 & \end{array}$ $\begin{array}{rl}758 & \\ \text{x}8 & \\ \hline 4 & (8\text{x}8=64)\\ 8 & (6+5=11;\ 11\text{x}8=88)\\ 120 & (8+7=15;\ 15\text{x}8=120)\\ \hline 12084 & \end{array}$ Added the renamed digit to the multiplicand before multiplying by the multiplier. If adding the renamed digit to the multiplicand produces a sum or 10 or more, and the child cannot multiply this larger number, various other methods of multiplying are attempted. The children neither showed the work to the right of the problem nor the partial products.
0%	0	33%	1	$\begin{array}{r}842\\ \text{x}7\\ \hline 849\end{array}$ $\begin{array}{r}539\\ \text{x}3\\ \hline 542\end{array}$ $\begin{array}{r}216\\ \text{x}6\\ \hline 222\end{array}$ Added instead of multiplying.

Table 34 (cont.)

26%	2	0%	0	842 x7 1094	758 x8 1164	376 x5 680	The "carried" number is added to the digit in the multiplier and the sum is placed in the answer. In this particular example the multiplication is correct in the ones and tens column.
12%	1	0%	0	81 842 x7 6524	22 758 x8 4249	83 376 x5 2330	In recording the product, the digits are somtimes reversed. In this particular case careless errors were also made.
12%	1	0%	0	842 x7 5884	539 x3 1597	216 x6 1266	In the tens column, the "carried" digit was not added to the product of the factors.
100%	8	100%	3				

Table 35

Systematic Errors in Multiplication Algorithm Level 5: One-digit Multiplier; Three and Four-digit Multiplicand; Renaming to Zeros

Normal		Handicapped		
Percentage	# of Errors	Percentage	# of Errors	Error
38%	8	32%	6	$\begin{array}{rrr}805 & 6070 & 403\\ \times4 & \times9 & \times6\\ \hline 3260 & 55530 & 2478\end{array}$ Added the renamed digit to the multiplier.
24%	5	37%	7	$\begin{array}{rrr}805 & 5117 & 6070\\ \times4 & \times8 & \times9\\ \hline 3280 & 43206 & 59430\end{array}$ Multiplied by the "carried" digit.
10%	2	11%	2	$\begin{array}{rrr}403 & 5007 & 906\\ \times6 & \times8 & \times7\\ \hline 2408 & 40006 & 6302\end{array}$ Did not add the "carried" digit after performing the necessary multiplication.
14%	3	5%	1	$\begin{array}{rrr}805 & 403 & 6070\\ \times4 & \times6 & \times9\\ \hline 3280 & 2468 & 99430\end{array}$ Added the "carried" number and multiplicand together. This new number is multiplied by the bottom factor and placed in the product. The ones column is multiplied correctly.
10%	2	0%	0	$\begin{array}{rrr}805 & 403 & 5007\\ \times4 & \times6 & \times8\\ \hline 1209 & 1072 & 13095\end{array}$ The mechanics of the vertical multiplication algorithm are used but instead of multiplying the digits are added. (In the middle example the digits in the ones column are multiplied.)

Table 35 (cont.)

0%	0	5%	1	$\begin{array}{r}6070\\ \text{x}9\\ \hline 30\\ 546\\ \hline 5490\end{array}$ $\begin{array}{r}403\\ \text{x}6\\ \hline 18\\ 24\\ \hline 258\end{array}$ $\begin{array}{r}5007\\ \text{x}8\\ \hline 056\\ 40\\ \hline 456\end{array}$	When multiplying a single digit times a three or four digit number, systematic errors are made in placing the product in the hundreds or thousands columns. The child wrote the partial products as shown in these examples.
0%	0	5%	1	$\begin{array}{r}403\\ \text{x}6\\ \hline 24018\end{array}$ $\begin{array}{r}5007\\ \text{x}8\\ \hline 400056\end{array}$ $\begin{array}{r}906\\ \text{x}7\\ \hline 63042\end{array}$	Places the entire product of the ones column in the answer instead of "carrying" it to the tens column.
0%	0	5%	1	$\begin{array}{r}805\\ \text{x}4\\ \hline 820\end{array}$ $\begin{array}{r}6070\\ \text{x}9\\ \hline 6070\end{array}$ $\begin{array}{r}403\\ \text{x}6\\ \hline 418\end{array}$	No multiplication is performed in the hundreds and thousands columns. The number in the multiplicand is placed in the answer.
4%	1	0%	0	$\begin{array}{r}805\\ \text{x}4^{2}\\ \hline 3420\end{array}$ $\begin{array}{r}6070\\ \text{x}9^{3}\\ \hline 5760\end{array}$ $\begin{array}{r}403\\ \text{x}6^{1}\\ \hline 2508\end{array}$	Renamed the first product by placing one of the digits to the right of the multiplier. This digit was then added to the other products.
100%	21	100%	19		

Table 36

Systematic Errors in Multiplication Algorithm
Level 6: Two-digit Multipliers With Zero in Ones Column; Two and Three-digit Multiplicands; With and Without Renaming

Normal		Handicapped					
Percentage	# of Errors	Percentage	# of Errors				Error
32%	5	70%	7	247 x20 287	149 x40 169	53 x20 103	Multiplies the number in the multiplicand by the number directly beneath it in the multiplier. Multiplication by zero can be either correct or contain the error n x 0 = n.
25%	4	10%	1	247 x20 494	26 x30 78	53 x20 106	Did not multiply by zero in the ones column. When the multiplier is a multiple of ten the answer is ten times too small.
25%	4	10%	1	18 x10 18 18 198	247 x20 247 494 5187	26 x30 26 78 806	The error n x 0 = n occurs since the multiplicand is used as the first partial product. Multiplication is performed correctly in the tens column.
0%	0	10%	1	247 x20 4840	26 x30 680	149 x40 4660	Did not add the "carried " number to the product of the multiplicand and multiplier.

Table 36 (cont.)

6%	1	0%	0	247 x20 2470	26 x30 00 26 260	53 x20 00 53 530	No multiplication is performed. The multiplicand is placed in the partial product. A zero is placed after the multiplicand in the answer.
6%	1	0%	0	247 x20 494000	26 x30 7800	53 x20 10600	When multiplying by a multiple of ten, too many zeros are annexed to the product.
6%	1	0%	0	247 x20 000 4814 48140	26 x30 00 618 6180	149 x40 0000 1636 16360	Did not rename. Places entire product in the answer.
100%	16	100%	10				

Table 37

Systematic Errors in Multiplication Algorithm
Level 7: Two-digit Multipliers; Three-digit Multiplicands With No Zeros in Either Factor; Multiple Renaming

Normal		Handicapped					
Percentage	# of Errors	Percentage	# of Errors				Error
67%	2	0%	0	346 x28 2768 692 9788	591 x46 3546 2364 36186	53 x74 212 371 4422	Added the partial product incorrectly.
0%	0	60%	3	346 x28 928	591 x46 1566	482 x64 1688	Did not cross-multiply. Multiplied the multiplicand by the digit directly below it. Digits in the hundreds column are either systematically mutliplied by the renamed ten placed above the column or consistently added.
33%	1	20%	1	346 x28 2768 6920 4248	47 x52 94 2350 2344	53 x74 212 3710 3502	Multiplied correctly, but instead of adding the partial products, the smaller one is subtracted from the larger one. This number is placed in the answer.
0%	0	20%	1	346 x28 2768	47 x52 94	591 x46 3546	Did not multiply by the tens column of the multiplier.
100%	3	100%	5				

Table 38

Systematic Errors in Multiplication Algorithm
Level 8: Two-digit Multipliers; Three-digit Multiplicand Containing Medial Zeros; Multiple Renaming

Normal		Handicapped					
Percentage	# of Errors	Percentage	# of Errors				Error
6%	1	0%	0	507 x32 1004 15010 16014	905 x46 5400 3600 41400	809 x52 1608 40050 41658	Did not add the "carried" number to the product of the multiplier and multiplicand.
0%	0	43%	3	507 x32 514	905 x46 930	809 x52 818	Multiplied the ones digit of the multiplicand by the ones digit of the multiplier. In the tens column, the "carried" number is placed in the answer. No multiplication is performed in the hundreds column. The number in the hundreds column of the multiplicand is placed in the hundreds column of the answer.
11%	2	14%	1	507 x32 1514	905 x46 3630	809 x52 4018	Did not multiply the hundreds digit of the multiplicand by the ones digit of the multiplier. Also did not multiply the ones and tens digit of the multiplicand by the tens digit of the multiplier.
39%	7	14%	1	507 x32 1034 1551 16544	905 x46 5490 3660 42090	809 x52 1638 4095 42588	When multiplying by zero, obtains "n" for the answer. (n x 0 = n).

Table 38 (cont.)

11%	2	29%	2	809 905 706 X52 x46 x47 1628 5580 5182 42050 36800 28840 43678 42380 80660	The "carried" number and the multiplier are multiplied together when the factor in the multiplicand is zero. (The second partial product was misplaced in the third example in this particular child's paper.)
6%	1	0%	0	507 905 809 x32 x46 x52 10014 54030 16018 17021 36020 40045 27035 90050 56063	A zero is incorrectly placed in the hundreds column of both partial products. The second partial product is misplaced. (A careless error was also made in the first example.)
6%	1	0%	0	507 905 706 x32 x46 x47 1034 5490 5012 1551 3660 28640 16544 417490 33652	When the factor in the multiplicand is a zero, the "carried" digit and the multiplier are added together. (A careless addition error was also made in adding the partial products in this child's paper.)
21%	4	0%	0	507 905 809 x32 x46 x52 1014 5430 1618 1521 3620 4045 2535 9050 5663	Misplaced the second partial product.
100%	18	100%	7		

Table 39

Systematic Errors in Multiplication Algorithm
Level 9: Three-digit Multipliers; Three-digit Multiplicands With No Zeros in Either Factor; Multiple Renaming

Normal		Handicapped				
Percentage	# of Errors	Percentage	# of Errors			Error
29%	2	0%	0	456 x251 456 22800 9120 32376	882 x198 7046 80380 8820 95246	Annexed incorrect number of zeros for the second and third partial products. (Careless errors were also made in the last example.)
0%	0	34%	1	456 x251 1056	627 x426 2482	Multiplied the number in the multiplicand by the number directly beneath it in the multiplier.
0%	0	33%	1	456 x251 456 2280 23256	882 x198 7056 7938 80436	Did not multiply by the hundreds digit of the multiplier. (A careless error was also made.)
14%	1	0%	0	456 x251 456 2280 219(should be 912) 23475	882 x198 7056 7938 288(should be 882) 8724	The digits in the third partial product are written from left to right, i.e., reversals. (Other careless errors were made.)

Table 39 (cont.)

29%	2	0%	0	456 x251 456 2280 912 113456	882 x198 7056 7938 882 154636	Added partial products incorrectly (many variations).
14%	1	0%	0	568 x348 4544 227200 170400 402144	882 x198 7056 733800 88200 829056	Misplaced the second partial product by annexing too many zeros.
14%	1	33%	1	627 x426 3762 1254 2408 257102	784 x143 2152 3136 784 111912	Did not add "carried" number at least one time in the problem.
100%	7	100%	3			

Table 40

Systematic Errors in Multiplication Algorithm
Level 10: Three-digit Multipliers With a Medial Zero; Three-digit Multiplicand With No Zeros; Multiple Renaming

Normal		Handicapped					
Percentage	# of Errors	Percentage	# of Errors				Error
30%	4	25%	1	456 x705 2280 0000 31920 34200	519 x402 1038 20760 21798		Annexed an insufficient number of zeroes for the last partial product or carelessly misplaced it.
0%	0	25%	1	436 x501 436 000 436	689 x307 4823 000 4823		Did not multiply by the hundreds digit of the multiplier. (A careless was also made.
0%	0	25%	1	224 x108 232	519 x402 2018	456 x705 2830	Multiplied the number in the multiplicand by the number directly beneath it in the multiplier.
0%	0	25%	1	456 x705 2980 28500 31480	519 x402 1438 20600 22038		Did not multiply the hundreds digit of the multiplier by each digit in the multiplicand. The first partial product is incorrectly derived by multiplying the ones digit times the multiplicand and adding the hundreds digit of the multiplier to the last multiplication. In the first example this is shown by: 456 x 5; 5 x 6 = 30; 5 x 5 = 25 + 3 = 28; 5 x 4 = 20 + 2 = 22 = 7 (from the hundreds digit of the multiplier) = 29.

Table 40 (cont.)

22%	3	0%	0	436 x501 436 0000 2180 2616	519 x402 1038 0000 2076 3114	Did not annex zeros to the last partial product.
8%	1	0%	0	456 x705 2280 45600 47880	519 x402 1038 51900 52938	No multiplication is performed with the digit in the hundreds column of the multiplier. Instead, two zeros are annexed to the multiplicand and it becomes the second partial product.
8%	1	0%	0	224 x108 17920 22400 40320	436 x501 4360 218000 222360	An extra zero is incorrectly annexed to the first partial product.
8%	1	0%	0	224 x108 1792 20000 21792	456 x705 2270 28000 30270	Did not multiply the ones and tens columns of the multiplicand by the hundreds column of the multiplier. (A careless error was also made.)
8%	1	0%	0	224 x108 1792 224 224 29132	519 x402 1038 519 2078 215028	Made the error, n x 0 = n. (A careless error was also made.)
8%	1	0%	0	456 x705 2280 0000 309200 311480	519 x402 1028 0000 207600 208628	Added a "carried" number incorrectly.
8%	1	0%	0	Many variations		Added the partial products incorrectly although the error in the addition is a random error.
100%	13	100%	4			

Figure 3
Example of Data Sheets

Level IV: Multiplication

NAME: ______________________________

GRADE: ____________

TEACHER'S NAME: ______________________

SCHOOL: ____________________________

DATE: ____________

842	539	216	758	376
x 7	x 3	x 6	x 8	x 5

Report No. 5

Systematic Errors in the Division Algorithm Normal and Handicapped Populations[1]

Linda S. Cox

This paper is the fourth in a series of reports from a larger research study to classify systematic errors that are made by normal and handicapped children and to determine the frequency of their occurrence. These errors are in the whole number operations in arithmetic. Results concerning the addition, subtraction, and multiplication algorithms have been reported in earlier papers (Cox, 1973a, 1973b, 1973c). The current report deals with the data on the division algorithm. Only a summary of the procedures and literature is presented here. More detailed information regarding the entire research study is presented in the paper on the addition algorithm.

Literature

A comprehensive review of the literature was conducted from 1900 through 1973 including a computer retrieval search of ERIC. The review focused on diagnosis, remediation, and error analysis in elementary school mathematics. Only four studies were located that dealt with errors in the division algorithm. Three of the four studies were reported by Grossnickle. In analyzing the long division algorithm with one-digit divisors, Grossnickle (1935) distinguished between two types of errors. He referred to "constant" errors and errors due to chance. Constant

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[1] This study was supported by grant number NS 05362 from the National Institute of Neurological Diseases and Stroke to the Bureau of Child Research, University of Kansas.

errors were defined as reoccurring incorrect responses to a specific number combination such as 7-5. He reported that 2% of the cases revealed constant errors in using the subtraction facts in the division algorithm and 5% of all cases revealed constant errors in using the multiplication facts in the division process. These results are concerned with the pupil's knowledge of subtraction or multiplication facts and do not relate to their ability to handle the algorithmic processes in division.

Grossnickle's second study (1936) still focused on the long division algorithm with a one-digit divisor. He reported all types of errors and disregarded the constancy of error. In this endeavor he reported 57 different errors and 13 questionable habits of work in grades 5-8. The 57 different errors were not described or illustrated but discussed in broad terms and categories of error.

In his third paper, Grossnickle (1939) examined the constancy of error in the division algorithm with a two-digit divisor. He limited his sample to fifth grade students who were learning the division process with a two-digit divisor. Results showed that most errors with two-digit divisors are due to chance. The 17 different kinds of errors resulting from zero were due entirely to chance. Most of the errors that were persistent resulted from faulty estimation and incorrect multiplication combinations. The errors with multiplication combinations again emphasize that many errors are made with the addition and multiplication combinations, a fact that Grossnickle has fairly well established.

Brueckner and Melbye (1940) followed this work by examining the relative difficulty of division problems. Establishing relative difficulty was done to facilitate placement of division topics in elementary school math textbooks. Relative difficulty was determined by examining the percentage of errors that were made on specific types of processes within the division algorithms for each mental age. Errors were not examined for constancy of occurrence.

In a few instances the literature discusses division errors within the analysis of other algorithmic errors. These discussions have not focused on systematic errors and have dealt with division errors in a broader context.

The present study examines only errors that are systematically mode in division and the frequency with which they occur in that consistent form.

Systematic Errors Defined

An error is labeled a systematic error when there is a repeatedly occurring incorrect response that is evident in a specific algorithmic computation. This incorrect process is apparent in three out of five problems of a given type. Systematic errors must be distinguished from random errors. Random errors give no evidence of a reoccurring incorrect process of thinking or recording.

Population

Normal and handicapped classrooms were chosen from the Shawnee Mission Public School system, a suburb of Kansas City. The handicapped population consisted of Learning Problem classrooms and Educable Mentally Retarded classrooms in the Shawnee Mission Public Schools, and classrooms for the emotionally disturbed at the University of Kansas Medical Center. A private and a parochial school also participated. The total sample size was 744. Descriptive information regarding the population is included in the earlier paper on the addition algorithm (Cox, 1973a).

Procedures

Ten levels of skill (see Table 41) were identified in the division algorithm. It should be noted that the levels of skill were not arranged in increasing order of difficulty. They were organized by the number of digits in the divisor and dividend and the inclusion or exclusion of zeros and remainders. Data sheets (see Fig. 4) were distributed each week beginning in March, 1973 through May, 1973.

Insert Table 41 & Fig. 4 (pp.106 & 124)here

Classroom teachers administered the data sheets. Two requirements had to be met before a child's paper was included for analysis. Those requirements were:

1. The child had been taught the basic facts that pertain to the algorithm being studied. The child may not score 100% on a test of the facts, but he knows most of them.

2. The child has been exposed to the computational skill levels of the algorithm although he may not be proficient in its use.

If a child did not meet the above two requirements his paper was not included for analysis. If a child did meet these requirements his paper was analyzed and classified in only one of the following categories.

Systematic error. The child missed at least three out of five problems, recording repeatedly the same incorrect type of response in the algorithm.

Random error. The child missed at least three out of five problems but a pattern or systematic incorrect process could not be detected.

Careless error. The child missed only one or two problems out of five. He basically knows how to work the algorithm.

No error. All five problems are correct.

Incomplete data sheet. The child did not work all five problems so that the data sheet could not be classified as one of the above types. Great effort was made to keep this category as small as possible during the data collection.

Results

Table 42 shows the frequency of all types of errors while Table 43 displays only the analysis of systematic errors. Percentages in Table 42 refer to children whose papers were classified as containing a specific systematic error.

Insert Table 42 (p.107) here

Table 42 shows that 6% of the papers that were analyzed were classified as containing a specific systematic error in division computations. The range for frequency of occurrence was from 1% to 17% of the time. Analyzing these percentages by skill level (Table 41) reveals that skill levels 6 and 9 had the highest percentages (17% and 10% respectively). These were the only skill levels to produce zeros in the quotient. It appears that for this particular sample of children two to three times more systematic errors were made when zeros were required in the answer. Inspection of Tables 49 and 52, reveal the types of systematic errors that children produced within these skill levels.

Table 43 shows only the percentage of systematic errors by grade levels. The mean percentages and grade levels are inversely proportional; however, specific percentages across grade levels are not all inversely proportional. Again it is for skill levels 6 and 9 where the relationship does not exist. For those two skill levels, the percentages rise and again fall from grade 4 to grade 6. For skill level 6 in grades 4, 5, and 6, the percentages are 12%, 17%, and 13%, respectively. They also rise between the intermediate and junior high classes. The percentages change from 20% for intermediate classes to 53% for junior high classes. The trend is similar for skill level 9 but the differences are much smaller.

It is in the grade 4 curriculum where division skills are initially emphasized. Fewer grade 4 children met requirements for inclusion in the study after skill level 6 even though skill levels 1-7 are included in several grade 4 textbooks. Inspection of Table 43 shows that for six of the skill levels, percentages were 8 or higher. This higher percentage may be due to the fact that the algorithmic process has just been introduced.

Insert Table 43 (p.108) here

The grade 4 students who did meet the requirements for skill levels 7-9 were on individualized contract programs. Therefore, one can assume that they may have had more proficiency with many of the division skills. This accounts for the 0% on skill level 7. However, the percentage of systematic errors rises again to 9% for skill level 8 and 13% for skill level 9.

Tables 44 through 53 define, illustrate, and tabulate all of the systematic errors.

Insert Tables 44-53 (pp.109-123) here

Discussion

The examples cited in Tables 44 through 53 are taken from children's data sheets. There were some deviations or individual variation on different children's work, but the systematic errors that are described were contained in at least three of the five problems. When many children made the same systematic error, as in the first error in Level 6, the most representative examples were illustrated. If only one child accounted for the systematic error, then the examples were taken from his paper.

The definition of errors are a combination of behavioral and mathematical statements. Mathematically, many of the systematic errors relate to a lack of understanding the division process, including the need to multiply and subtract; incorrect formation of partial dividends; incorrect use of partial quotients; the function of place value; and the concept of remainders.

Many of the random errors were similar to the systematic errors. In considering both populations, 11% of the total errors were classified as random responses. Many of the random error responses were in forming the new partial dividend. Others involved not "bringing down the next digit."

Random error responses for skill levels 7-10 were higher. These skill levels all involved two-digit divisors. Many of the incorrect responses for these skill levels were due to errors with the multiplication process. However, none of the errors were specific enough nor occurred often enough to be classified as systematic.

Incorrect responses in the division algorithm produced some unique systematic errors. For example, one error in level 5 was:

$$\begin{array}{r} 1221\text{ r.}3 \\ 4\overline{)507} \\ 4 \\ \overline{10} \\ 8 \\ \overline{87} \\ 84 \\ \overline{3} \end{array} \qquad \begin{array}{r} 233\text{ r.}2 \\ 3\overline{)702} \\ 6 \\ \overline{10} \\ 9 \\ \overline{92} \\ 90 \\ \overline{2} \end{array} \qquad \begin{array}{r} 126\text{ r.}3 \\ 7\overline{)905} \\ 7 \\ \overline{20} \\ 14 \\ \overline{45} \\ 42 \\ \overline{3} \end{array}$$

This child did not perform the second subtraction correctly but he understands the concept of division. If his error were explained to him, he probably would not continue making this error. Since he appears to understand the concept of division, retention regarding this error might be quite good. In this particular case, remediation might be fairly simple. However, other cases are very difficult. Techniques need to be developed that remediate specific difficult systematic errors. These techniques should then be tested for effectiveness.

Children in the introductory stages of learning division processes appear to have the most difficulty. Percentages were highest for fourth grade and the intermediate handicapped classes. However, even in sixth grade 3% of the children still made systematic errors and in junior high handicapped classrooms, 13% still made these errors.

Overall, the children in handicapped classrooms appear to have three times the percentage of systematic errors than children in regular classrooms. Using Table 43 and averaging, separately, the percentages for normal and handicapped classrooms the figure is 5% for the regular classrooms compared to an average of 17% for the handicapped classrooms. These percentages are sufficiently large to merit the attention of teachers of children with learning problems and to direct the attention of future research on the remediation of systematic errors.

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Table 41

Levels of Skill in Division*

Level 1:	One-digit divisor; two-digit dividend; no remainders.	$4\overline{)48}$
Level 2:	One-digit divisor; two-digit dividend; with remainders.	$5\overline{)48}$
Level 3:	One-digit divisor; three-digit dividends; no remainders.	$5\overline{)455}$
Level 4:	One-digit divisor; three-digit dividends; with remainders.	$5\overline{)346}$
Level 5:	One-digit divisor; three-digit dividends with zeros; with and without remainders.	$5\overline{)608}$
Level 6:	One-digit divisor; three-digit dividends that produce zeros in tens column in quotient; with and without remainders.	$4\overline{)436}$
Level 7:	Two-digit divisors; three-digit dividends; no zeros; no remainders.	$35\overline{)735}$
Level 8:	Two-digit divisors; four-digit dividends; with remainders.	$36\overline{)5438}$
Level 9:	Two digit divisors; four-digit dividends producing zeros in quotient; with remainders.	$26\overline{)5446}$
Level 10:	Three-digit divisors; five-digit dividends; with remainders; complex long division.	$384\overline{)46,590}$

*These levels are not necessarily in order of increasing difficulty. They are organized by the number of digits in the divisor and dividend, the inclusion or exclusion of zeros, in the dividend and quotient, and the existence or absence of remainders.

Table 42

All Types of Errors for Both Populations for All Grades in the Division Algorithm

Skill Level	# of Papers Analyzed	Systematic Errors	Random Errors	Careless Error	No Error	Incomplete Data Sheet
		3 out of 5	3 out of 5	1 or 2 out of 5		
1	348	6%	6%	16%	64%	8% = 100%
2	315	6%	6%	20%	64%	4% = 100%
3	271	1%	9%	30%	54%	6% = 100%
4	257	2%	10%	40%	44%	4% = 100%
5	283	7%	5%	37%	48%	3% = 100%
6	286	17%	3%	17%	61%	2% = 100%
7	201	1%	9%	32%	55%	3% = 100%
8	207	4%	12%	44%	34%	6% = 100%
9	153	10%	18%	39%	25%	8% = 100%
10	82	2%	30%	37%	10%	21% = 100%
Average Per Skill Level	240	6%	11%	31%	46%	6% = 100%

Table 43

Percentage of Systematic Errors in Division Algorithm by Population, Grade, and Skill Level

Levels of Skill	Normal Population Grades					Handicapped Population Classroom			Both Populations
	2	3	4	5	6	Primary	Intermediate	Jr.Hi.	
1	*	11%	12%	3%	2%	*	30%	6%	6%
2	*	*	12%	3%	1%	*	20%	12%	6%
3	*	*	3%	0%	0%	*	14%	0%	1%
4	*	*	2%	1%	2%	*	25%	8%	2%
5	*	*	8%	7%	4%	*	100%	5%	7%
6	*	*	12%	17%	13%	*	20%	53%	17%
7	*	*	0%	1%	0%	*	*	0%	1%
8	*	*	9%	4%	3%	*	*	9%	4%
9	*	*	13%	14%	3%	*	*	22%	10%
10	*	*	0%	0%	3%	*	*	13%	2%
Average % by Grade Level	_	_	7%	5%	3%	*	21%	13%	6%
Total # of Papers Analyzed/Grade		27	411	845	888	_	38	194	2403
Average # of Papers Analyzed/Skill Level/Grade Level	_	27	41	85	89	_	6	19	

*Classrooms were not tested because no child could meet the requirements for the study.

Table 44

Systematic Errors in the Division Algorithm:
Level 1 - One-digit Divisor; Two-digit Dividend; No Remainders

Number of Errors		
Normal	Handicapped	Error
3	2	$\begin{array}{r} 11 \\ 4\overline{)56} \end{array}$ $\begin{array}{r} 34 \\ 2\overline{)78} \end{array}$ $\begin{array}{r} 11 \\ 5\overline{)85} \end{array}$ Each digit of the dividend is divided separately by the divisor without performing the subsequent operations of multiplication, subtraction, and forming the next partial dividend.
0	2	$\begin{array}{r} 1 \\ 6\overline{)90} \\ \underline{6} \\ 0 \end{array}$ $\begin{array}{r} 3 \\ 2\overline{)78} \\ \underline{6} \\ 8 \end{array}$ $\begin{array}{r} 1 \\ 5\overline{)85} \\ \underline{5} \\ 5 \end{array}$ Division in the tens column is correct but no subtraction is performed with the dividend and the product. No division is performed in the ones column.
0	1	$\begin{array}{r} 12 \\ 4\overline{)56} \\ \underline{4} \\ 16 \end{array}$ $\begin{array}{r} 31 \\ 3\overline{)96} \\ \underline{9} \\ 6 \end{array}$ Underestimated the ones column of the quotient.
3	0	$\begin{array}{r} 1 \\ 4\overline{)56} \\ \underline{4} \\ 16 \\ \underline{16} \\ 0 \end{array}$ $\begin{array}{r} 3 \\ 3\overline{)96} \\ \underline{9} \\ 06 \\ \underline{6} \end{array}$ $\begin{array}{r} 1 \\ 6\overline{)90} \\ \underline{6} \\ 30 \\ \underline{30} \\ 0 \end{array}$ Did not divide in the ones column.
2	0	$\begin{array}{r} 11r.3 \\ 4\overline{)56} \end{array}$ $\begin{array}{r} 10r.3 \\ 6\overline{)90} \end{array}$ $\begin{array}{r} 34r.1 \\ 2\overline{)78} \end{array}$ Treats each digit of the dividend as single-digit numbers. Does not show any work below the dividend. Each digit is divided separately by the divisor. The remainder is found by adding two subtracted numbers, one from the first division process and the second subtracted figure from the second division process.
2	0	$\begin{array}{r} 9 \\ 6\overline{)90} \\ \underline{56} \\ 44 \end{array}$ $\begin{array}{r} 9 \\ 2\overline{)78} \\ \underline{18} \\ 60 \end{array}$ $\begin{array}{r} 9 \\ 5\overline{)85} \\ \underline{45} \\ 45 \end{array}$ Incorrectly estimated the quotient digit. Failed to divide the divisor into the first number of the dividend. Perseverate activity in choosing nine for the quotient.

Table 44 (cont.)

1	0	10 30 10 4)56 3)36 6)90	The quotient is a multiple of ten. The operations of multiplication, subtraction, and forming the next partial dividend are omitted.
1	0	13r.4 14r.6 4)56 6)90 52 84 4 6	Estimates the quotient as a two-digit number. Correctly multiplies the divisor times the two-digit quotient and correctly subtracts. The error occurs in underestimating the initial two-digit quotient and not realizing that the remainder is too large.
1	0	95 9995 4)56 3)96 -36 -27 20 69 -20 -27 42 -27 15 15	The child uses the subtractive algorithm for division but fails to add the partial quotients to derive the total quotient. For example, the first problem should have been worked: 14 4)56 -36 9 20 -20 +5 0 14
1	0	2 2 4)56 3)96 8 3 136 126	Adds instead of subtracts to obtain the partial dividend.
1	0	1r.6 3r.3 4)56 3)96 4 3 16 66 16 66 6 6	The first partial quotient is incorrectly placed in the ones column instead of the tens column. After multiplying, subtracting, and forming the next partial dividend, no division

Table 44 (cont.)

			is performed. The remainder is the digit in the ones column of the dividend. We could predict that in another problem, the child would do the following: $\begin{array}{r}1r.7\\3\overline{)57}\\3\\\hline 27\\27\\\hline 7\end{array}$
$\frac{1}{16}$	$\frac{0}{5}$	$\begin{array}{r}13\\6\overline{)9{}^{3}0}\end{array}$ $\begin{array}{r}35\\2\overline{)7{}^{1}8}\end{array}$	With the ten's digit of the dividend, the child correctly divides, multiplies, subtracts, and forms the next partial dividend. Incorrectly estimates the second partial dividend and does not multiply, subtract, nor form the remainder.

Table 45

Systematic Errors in the Division Algorithm:
Level 2 – One-digit divisor; Two-digit Dividend; With Remainders

Number of Errors				
Normal	Handicapped			Error
3	1	15 5)79	12 7)86	Estimated the quotient accurately but failed to show the multiplication, subtraction or formation of remaining partial dividends. Thus failed to show the remainder.
0	2	15 5)79 5 29	16 2)33 2 13	Did not perform the second multiplication causing the second subtraction not to be performed and thus failing to show a remainder.
1	1	21 4)95	11 2)33	Divided each digit of the dividend separately by the divisor. Did not multiply, subtract, or form a new partial dividend.
2	0	11r.24 5)79 55 24	21r.11 4)95 84 11	Perseverated in estimating the quotient as a multiple of eleven and failed to catch the error when the remainder is larger than the divisor.
0	1	15 5)79 5 29 25	12 7)86 7 16 14	Worked the algorithm correctly until the last step. Did not perform the last subtraction, thus failing to show a remainder.
0	1	2 4)95 8 15 12 3 3 0	1 2)33 2 13 10 3 3 0	Did not record the quotient digit for the ones column. Subtracted the remainder from itself to leave a zero for the remainder.

Table 45 (cont.)

1	0	2 5)79	5 4)95	No division is performed. The divisor is subtracted from the number in the tens column of the dividend and this answer is placed in the quotient. No operation is performed in the ones column.
1	0	17r.2 7)86 7 16 14 2	28r.3 4)95 8 15 12 3	When dividing in the ones column, the child performs the operation of addition instead of multiplication.
1	0	74 5)79	91 4)95	Subtracts the divisor from the dividend and places this answer in the quotient.
1	0	1 7)86 7 16 16 0	2 4)95 8 15 15 0	No division is performed in the ones column.
1 11	0 6	11 5)79 5 29 5 4	21 4)95 8 15 4 1	In dividing the second partial dividend, the quotient is underestimated. Subtraction is performed only in the units column.

Table 46

Systematic Errors in the Division Algorithm: Level 3 - One-digit Divisor; Three-digit Dividends; No Remainders

Number of Errors		
Normal	Handicapped	Error
1	0	32 64 8)232 6)342 24 36 12 22 16 24 4 2 Overestimates the quotient and does not catch the error because subtraction is always performed. If the subtrahend is the larger number, the smaller minuend is subtracted from it. Thus, the child can always subtract to form another partial dividend and can continue to divide.
1	0	91 72 5)465 4)292 Explanation: 46 ÷ 5 = 9 29 ÷ 4 = 7 5 ÷ 5 = 1 4 ÷ 2 = 2 The three-digit dividend is seen as two separate numbers and division is performed separately on the two numbers. Did not multiply, subtract, nor form new partial dividends. (The child did not show his work on the paper.)
$\frac{0}{2}$	$\frac{1}{1}$	713r.7/3 573r.6/4 7)494 6)342 Explanation: 49 ÷ 7 = 7 34 ÷ 6 = 5 9 ÷ 7 = 1 44 ÷ 6 = 7 24 ÷ 7 = 3 22 ÷ 6 = 3 3 = r. 7/3 4 = r. 6/4 Divided the divisor into the tens digit twice, once in the first division process and secondly as the single tens digit. (No work was shown on the child's paper.)

Table 47

Systematic Errors in the Division Algorithm:
Level 4 - One-digit Divisor; Three-digit Dividends; With Remainders

Number of Errors		Error		
Normal	Handicapped			
1	3	73 $5\overline{)346}$ $\underline{35}$ 16 $\underline{15}$ 1	92 $4\overline{)470}$ $\underline{36}$ 11 $\underline{8}$ 3	Although random errors were made, the digits in the quotient were over- or underestimated. This error was not detected in the subtraction steps of the algorithm.
1	0	117r.8 $4\overline{)470}$ $\underline{4}$ 7 $\underline{4}$ 30 $\underline{28}$ 8	151r.7 $7\overline{)961}$ $\underline{7}$ 36 $\underline{35}$ 11 $\underline{7}$ 7	Errors in subtraction are the consistent reoccurring problem.
1	0	7r.1 $5\overline{)346}$ $\underline{5}$ 24 $\underline{35}$ 1	4r.3 $8\overline{)255}$ $\underline{8}$ 65 $\underline{32}$ 3	The divisor is used as the first number placed below the dividend. Subtraction is always performed by taking the smaller number from the larger one. The division process was not completed with all of the digits in the dividend.
3	3			

Table 48

Systematic Errors in Division Algorithm:
Level 5: One-digit Divisor; Three-digit Dividends With Zeros; With and Without Remainders

Number of Errors Normal	 Handicapped	Error		
5	3	101 5)608	20 3)702	Divides the divisor into each digit separately without multiplying, subtracting, or forming any new partial dividends.
3	0	121r.5/3 5)608 5 10 10 8 5 3	126r.4/3 4)507 4 10 8 27 24 3	Inverted the remainder.
2	0	120r.2 4)507 4 10 8 2 0 2	230r.1 3)702 6 10 9 1 0 1	No division is performed in the ones column because the number in the ones column is not brought down. A zero is placed in the answer.
2	0	1206 4)507 480 27 24 3	2304 3)702 690 12 12	Did not add partial quotients, but placed the second partial quotient to the right of the first one. (In the second example at left, the quotient should be 230 + 4 = 234.)

Table 48 (cont.)

1	0	1221r.3 4)507 4 10 8 87 84 3	2330r.2 3)702 6 10 9 92 90 2	Did not correctly perform the second subtraction. Thus, the second partial dividend is incorrect.
1	0	24 3)702 6 12	13 7)905 7 25 21	When a zero occurs in a medial position in the dividend, the zero is ignored and not brought down to form a new partial dividend. Instead, the next number to the right is brought down.
1	0	122r.1 4)507 4 10 8	123 7)905 7 20 14 105	The ones column of the quotient is incorrectly estimated because of subtraction errors.
15	3			

Table 49

Systematic Errors in Division Algorithm:
Level 6: One-digit Divisor; Three-digit Dividends That Produce Zeros In Tens Column in Quotient; With and Without Remainders

Number of Errors				
Normal	Handicapped	Error		
28	9	19 4)436	47 2)814	Errors occur because a zero is not placed in the tens place in the quotient. This occurs when a digit in the dividend is brought down and a division can not occur because the divisor is too large. Then the zero which should be placed in the quotient is omitted and the next division is computed. The result is quotients with zeros missing in the middle term.
9	1	101r.2 4)436	402 2)814	If one of the medial digits in the quotient is a zero, the next two-digit partial dividend is not formed. Instead the next digit in the dividend to the right is used and the divisor is divided into it separately.
1	0	190 4)436 4 36 36 0	470 2)814 8 14 14 0	The zero which should be placed in the tens column of the quotient is placed in the ones column.
1	0	104 4)436 4 3 4 16	104r.3 5)533 5 3 5 23	Does not correctly form the first partial dividend when the first number that is brought down is too small. Does not bring down a second number to form the correct partial dividend. A zero is correctly placed in the quotient for the number that is brought down but the multiplication of zero times the divisor produces the divisor. Subtraction is accomplished by subtracting the smaller number from the larger number.
39	10			

Table 50

Systematic Errors in Division Algorithm
Level 7: Two-digit Divisors; Three-digit Dividends; No Zeros; No Remainders

Number of Errors				
Normal	Handicapped			Error
1	0	121 5/3 5)608 5 10 10 8 5 3	126 4/3 4)507 4 10 8 27 24 3	Inverted the fraction that represented the remainder.

Table 51

Systematic Errors in Division Algorithm
Level 8: Two-digit Divisors; Four-digit Dividends; With Remainders

Number of Errors		Error		
Normal	Handicapped			
2	0	403r.40 65)2835 260 23 0 235 195 40	402r.35 55)2345 220 14 0 145 110 35	Incorrectly placed the first digit of the quotient which resulted in placing a zero in the tens column of the quotient.
1	0	$43\frac{65}{30}$ 65)2835 260 235 195 30	$23\frac{73}{59}$ 73)1738 146 278 219 59	Inverted the fraction.
1	0	43 65)2835	42 55)2345	Did not indicate any remainders. Placed the digits in the quotients in the wrong columns.
1	0	1279r.1 65)~~28~~35 65 62 5 1	12139r.3 55)~~23~~45 55 78 5 3	Renamed the dividend and wrote the renamed numerals in the quotient. Used the divisor and last digit of the dividend in the work under the division sign.
3 8	1 1	430r.40 65)2835 260 235 195 40	420r.35 55)2345 220 145 110 35	Unnecessarily placed a zero in the ones column of the quotient.

Table 52

Systematic Errors in Division Algorithm
Level 9: Two-digit Divisors; Four-digit Dividends Producing Zeros in the Quotient; With Remainders

Number of Errors				
Normal	Handicapped	Error		
9	1	29 r.14 26)5448 52 248 234 14	19 r.43 48)5275 48 475 432 43	Did not place a zero in the quotient to function as a place holder where a division could not occur. In these cases, a zero is missing in the tens column of the quotient.
2	0	200r.8 26)5448 52 24 0 08 8	101 r.27 48)5275 48 47 0 75 48 27	Subtraction errors and subsequent incorrect formation of partial dividends leads to incorrect quotients in the ones column.
2	0	290r.14 26)5448 52 248 234 14	190r.43 48)5275 48 475 432 43	Did hot place a zero in the tens column of the quotient, incorrectly placed the ones digit in the tens column, and placed a zero in the ones column to fill an empty space.
1	0	208r.40 26)5448 52 248 208 40	108r.91 48)5275 48 475 384 91	Underestimates the digit for the ones column of the quotient and did not catch this error following the subtraction process.

Table 52 (cont.)

$\frac{0}{14}$	$\frac{1}{2}$	2724 26)5448 4 14 14 4 4 8 8	1318r.3 48)5275 4 12 12 7 4 35 32 3	Divides by only one of the digits of the divisor.

Table 53

Systematic Errors in Division Algorithm
Level 10: Three-digit Divisors; Five-digit Dividends; With Remainders; Complex Long Division

Number of Errors				
Normal	Handicapped	Error		
1	0	32)33285 13200 100 20085 13200 100 6885 2640 20	263)72859 26300 100 46559 26300 100 20259	Didn't finish the problem.
0 1	1 1	33142r.1 132)33285 3 03 3 02 2 08 8 05 4 1	41235r.3 246)82543 8 02 2 05 4 14 12 23 20 3	Divides by only one of digits of the divisor.

Figure 4
Example of Data Sheets

Level II, Division

NAME:______________________________

GRADE:____________

TEACHER'S NAME:______________________

SCHOOL:_____________________________

DATE:______________

$5\overline{)79}$ $7\overline{)86}$ $4\overline{)95}$ $2\overline{)33}$ $6\overline{)77}$

LINDA S. COX

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Instructor, University of Kansas, 1967-70
Assistant Professor, California State University, Hayward, 1970-72
Research Trainee in Communication Disorders, Bureau of Child Research, University of Kansas, 1972-1974
Associate Professor, Avila College, Kansas City, 1974-

Fields of Present Major Scientific Interest

An experimental analysis of learning problems in education, specifically in mathematics and reading.

Supplemental Information

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